A Perspective On

# Annular Khovanov Homology

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Perspectives on Quantum Link Homology Theories

2021 August 9-13 University of Regensburg

**Roadmap** (assuming perfect weather and travel conditions)

Lect. 1

Notation, definition, computation

History and Motivation: categorified skein module

Application: Periodic Links

Lect. 2

Structure of AKh:

$\mathfrak{sl}_2(\mathbb{C})$ action

Annular Khovanov-Lee Homology

Lect. 3

Distilled Numerical Invariants: annular Rasmussen invariants

Annular filtrations on other Khovanov Homologies

Applications to

knot concordance

transverse knots

braids as MCG($\mathbb{D}^2$, some points)

Lect. 4

Braid detection results

AKh and Floer theories

Spectral Sequences

Knot detection results

## An Important Remark

This is nowhere near a complete perspective! We'll focus on

- understanding the definition and structure of AKh
- a few applications to low-dimensional topology

for the benefit of graduate student workshop attendees

I will skip some major developments (such as [Beliakova-Putyra-Wehrli]'s quantum annular link homology). However, I am compiling a list of references in the text version of this lecture series.

It is very important to me to represent everyone's work fairly and accurately.

If you have done work related to annular Khovanov homology that I have not mentioned, please feel free to inform me of your perspective so that I can include your work in the compilation.

Thank you for your time!

# Review of Khovanov Homology + notation

$L \subset S^3$ oriented link

$D(L)$ on $S^2$ link diagram

$n$ # crossings

$n_+$ # positive crossings

$n_-$ # negative crossings

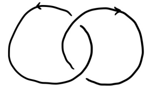
Cube of resolutions:

- vertices: $u \in \{0,1\}^n$

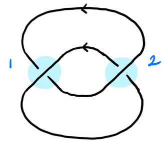
  $|u| =$ # of "1"s in string $u$


- edges: $u \to w$ whenever $\underbrace{u <_1 w}_{\text{increment exactly one bit from 0 to 1}}$

Running Example    $L =$ Hopf link



D(L)

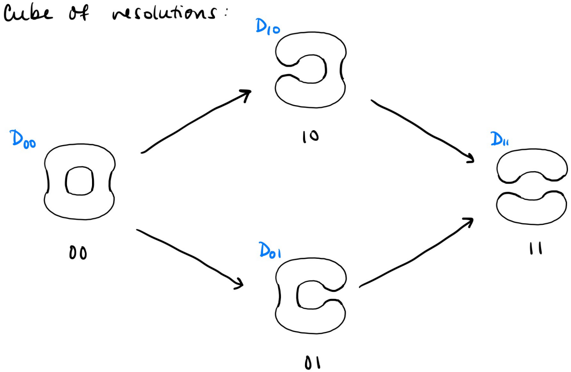



$n = 2$

$n_+ = 2$

$n_- = 0$

Cube of resolutions:



# Review of Khovanov Homology + notation

| | |
|---|---|
| $n$ | # crossings |
| $n_+$ | # positive crossings |
| $n_-$ | # negative crossings |
| $R$ | ground ring (eg. $\mathbb{Z}, \mathbb{C}, \mathbb{F}_2$) |

① generators of $Kc(D)$ ("Khovanov chains") as free $R$-module

<u>Kauffman states</u> : labelings of each $D_u$ with a $+$ or $-$ on each planar circle

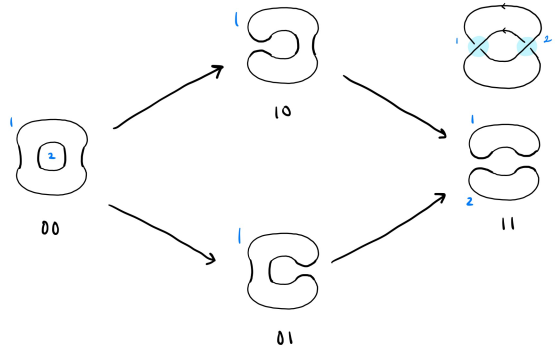
i.e.

$Kg$ "Khovanov generators" = pure tensors in the symbols $x_+$ and $x_-$, after choosing an ordering of the circles

i.e.

$D_u \rightsquigarrow V^{\otimes \,\#\text{ circles in } D_u}$

where $V = Rx_+ \oplus Rx_-$




$$\begin{tikzcd}
 & R\left\langle \begin{matrix} x_+ \\ x_- \end{matrix} \right\rangle \arrow[dr] & \\
R\left\langle \begin{matrix} x_+ \otimes x_+ \\ x_+ \otimes x_- \\ x_- \otimes x_+ \\ x_- \otimes x_- \end{matrix} \right\rangle \arrow[ur] \arrow[dr] & & R\left\langle \begin{matrix} x_+ \otimes x_+ \\ x_+ \otimes x_- \\ x_- \otimes x_+ \\ x_- \otimes x_- \end{matrix} \right\rangle \\
 & R\left\langle \begin{matrix} x_+ \\ x_- \end{matrix} \right\rangle \arrow[ur] &
\end{tikzcd}$$

# Review of Khovanov Homology + notation

| | |
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| $n$ | # crossings |
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| $n_-$ | # negative crossings |
| $R$ | ground ring (eg. $\mathbb{Z}, \mathbb{C}, \mathbb{F}_2$) |
| $V$ | $= Rx_+ \oplus Rx_-$ |

② gradings on distinguished generators $Kg$

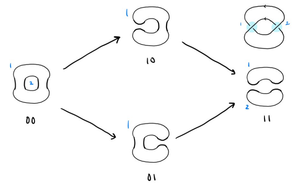
let $x \in Kg(D)$ at $D_u$, $u \in \{0,1\}^n$

homological grading $gr_h$

$$gr_h(x) = |u| - n_-$$

quantum grading $gr_q$

$$gr_q(x) = |u| + \#(x_+) - \#(x_-) + n_+ - 2n_-$$




quantum gradings in red

$$\begin{array}{ccccc}
gr_h = 0 & & gr_h = 1 & & gr_h = 2 \\[6pt]
R\left\langle \begin{array}{c} x_+\otimes x_+ \\ x_+\otimes x_- \\ x_-\otimes x_+ \\ x_-\otimes x_- \end{array} \right\rangle \begin{array}{c} 4 \\ 2 \\ 2 \\ 0 \end{array}
& \begin{array}{c} \nearrow \\ \\ \searrow \end{array} &
\begin{array}{c} R\left\langle \begin{array}{c} x_+ \\ x_- \end{array} \right\rangle \begin{array}{c} 4 \\ 2 \end{array} \\ \oplus \\ R\left\langle \begin{array}{c} x_+ \\ x_- \end{array} \right\rangle \begin{array}{c} 4 \\ 2 \end{array} \end{array}
& \begin{array}{c} \searrow \\ \\ \nearrow \end{array} &
R\left\langle \begin{array}{c} x_+\otimes x_+ \\ x_+\otimes x_- \\ x_-\otimes x_+ \\ x_-\otimes x_- \end{array} \right\rangle \begin{array}{c} 6 \\ 4 \\ 4 \\ 2 \end{array} \\[6pt]
Kc^0(D) & \longrightarrow & Kc^1(D) & \longrightarrow & Kc^2(D)
\end{array}$$

$$gr_q = 0 + 0 - 2 + 2 - 2\cdot 0 = 0$$

# Review of Khovanov Homology + notation

## ③ Differential $d_{Kh}$

If $u <_1 w$, then $D_u$ and $D_w$ differ at exactly one crossing. 2 possibilities:


$V \otimes V \xrightarrow{m} V$

$x_+ \otimes x_+ \longrightarrow x_+$

$x_+ \otimes x_- \longrightarrow x_-$, $\quad x_- \otimes x_+ \longrightarrow x_-$

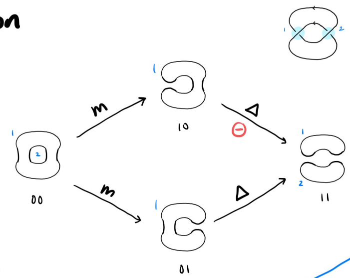
$x_- \otimes x_-$

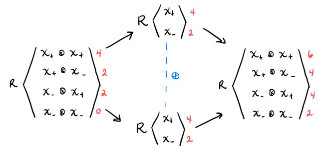

$V \xrightarrow{\Delta} V \otimes V$

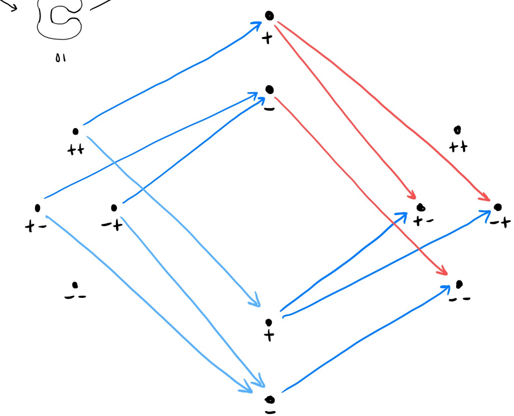
$x_+ \otimes x_+$

$x_+ \longrightarrow x_+ \otimes x_- \quad x_- \otimes x_+$

$x_- \longrightarrow x_- \otimes x_-$




⊖ so that the square anticommutes. ($d^2 = 0$)

Then extend by the identity map on all nonaffected circles in $D_u$ and $D_w$.

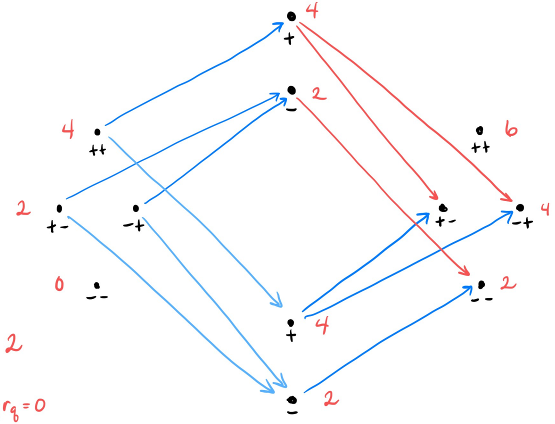
eg. $V \otimes V \otimes V \otimes V \otimes V \xrightarrow{id \otimes id \otimes m \otimes id} V \otimes V \otimes V \otimes V$

# Review of Khovanov Homology + notation : Summary

$$Kh(L) = H^*\Big( \big(Kc(D), d_{Kh}\big) \Big)$$

- $Kc^i(D) = \bigoplus_{|u|=i} V^{\otimes \#\text{ circles in } D_u}$
- $d_{Kh}$ has $(gr_h, gr_q)$ bigrading $(1,0)$

Kh(L) does not depend on the choice of diagram D(L)!




eg. (Running example) $L =$ Hopf link, $R = \mathbb{Z}$

$$Kh^0(D) = \frac{\ker}{\mathrm{im}} = \frac{\mathbb{Z}\langle x_+ \otimes x_- - x_- \otimes x_+,\ x_- \otimes x_- \rangle}{0} \cong \mathbb{Z} \oplus \mathbb{Z}$$

$gr_q = 2$

$gr_q = 0$

$$Kh^1(D) = \frac{\ker}{\mathrm{im}} = \frac{\mathbb{Z}\langle x_+^{10} - x_+^{01},\ x_-^{10} - x_-^{01} \rangle}{\mathbb{Z}\langle x_+^{10} - x_+^{01},\ x_-^{10} - x_-^{01} \rangle} \cong 0$$

$gr_q = 6$ $\quad gr_q = 4$

$$Kh^2(D) = \frac{\ker}{\mathrm{im}} = \frac{\mathbb{Z}\langle x_+ \otimes x_+,\ x_+ \otimes x_-,\ x_- \otimes x_+,\ x_- \otimes x_- \rangle}{\mathbb{Z}\langle x_+ \otimes x_- + x_- \otimes x_+,\ x_- \otimes x_- \rangle} \cong \mathbb{Z} \oplus \mathbb{Z}$$

# Annular Khovanov Homology

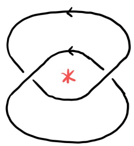
Annular link:

$$L \subset \underbrace{\left(\mathbb{R}^2 \setminus \{0\}\right)}_{= A \text{ (annulus)}} \times [0,1]$$

$$\cong S^3 \setminus \underbrace{\text{Unknotted } S^1}_{z\text{-axis} \,\cup\, \{\infty\}}$$

eg. $L$ = Hopf link as $\widehat{\sigma_1^{\,2}}$ ← braid closure

D(L)

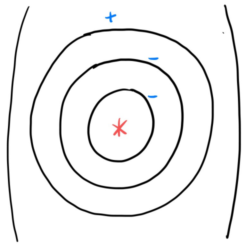


New annular grading on Kg:

3 nontrivial circles

(circles that separate 0 and ∞)

$gr_K$

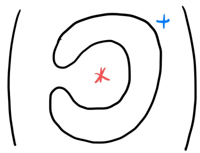



$= 1-1-1 = -1$

1 trivial circle

(circle that does not separate 0 from ∞)

$gr_K$

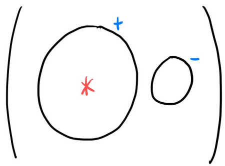



$= 0$

1 nontrivial circle

1 trivial circle

$gr_K$




$= 1 + 0 = 1$

# Annular Khovanov Homology

① generators $= Kg$ (same as $Kh$)

② gradings

homological $gr_h$

quantum $gr_q$

winding number $gr_k$ :

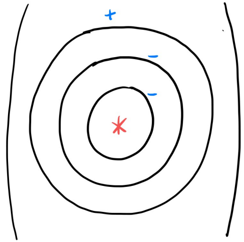
- $x_+$ on nontrivial circle $= v_+$ $\quad gr_k = +1$
- $x_-$ on nontrivial circle $= v_-$ $\quad gr_k = -1$
- $x_\pm$ on trivial circle $= w_\pm$ $\quad gr_k = 0$

③ differential $d_{AKh}$

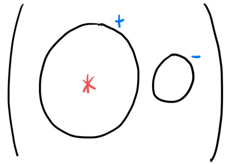
$=$ components of $d_{Kh}$ that preserve $gr_k$

$$d_{Kh} = \underbrace{d_0}_{d_{AKh}} + d_{-2}$$

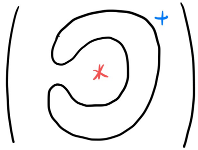
New annular grading on $Kg$ :

$gr_k$  $= 1-1-1 = -1$



$gr_k$  $= 1 + 0$



$gr_k$  $= 0$

# Annular Khovanov Homology

- $x_+$ on nontrivial circle $= v_+$ $\quad gr_k = +1$
- $x_-$ on nontrivial circle $= v_-$ $\quad gr_k = -1$
- $x_\pm$ on trivial circle $= w_\pm$ $\quad gr_k = 0$

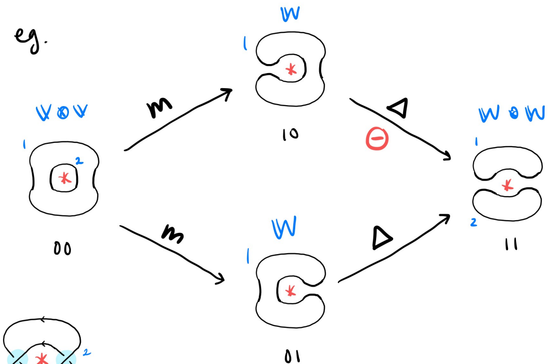
Replace our $R$-module $V = R\langle x_+, x_- \rangle$:

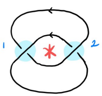
$\mathbb{V} = R\langle v_+, v_- \rangle$

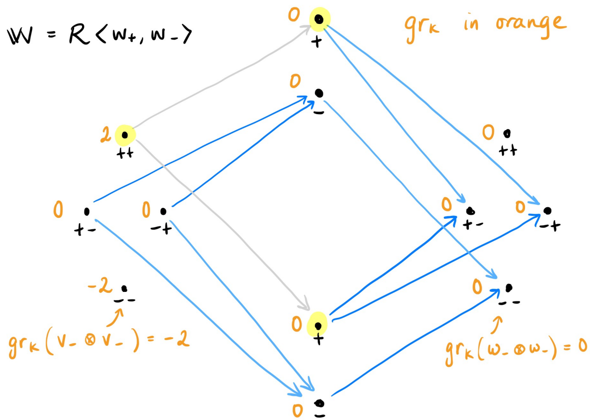
$\mathbb{W} = R\langle w_+, w_- \rangle$

$d_{AKh}$ = components of $d_{Kh}$ that preserve $gr_k$

eg.



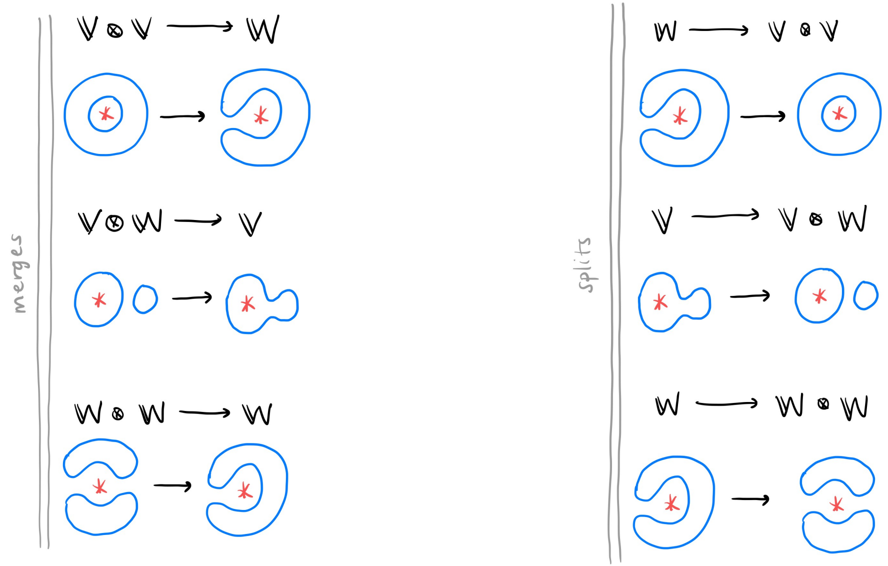





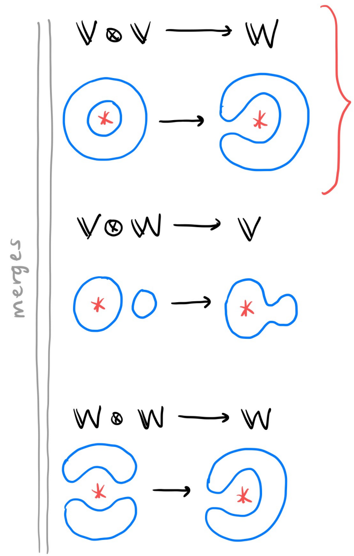

ex. Finish computing $AKh(\widehat{\sigma_1^2})$ for $R = \mathbb{C}$.
Make sure to record the $(gr_h, gr_q, gr_k)$ trigrading of each dimension.

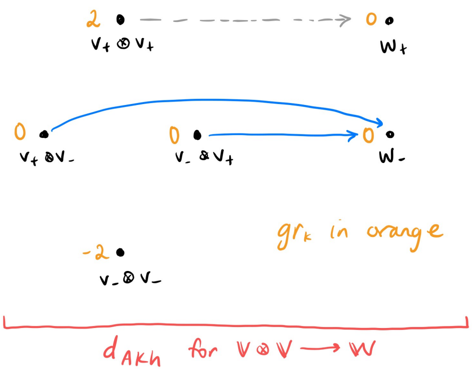
# Annular Khovanov Homology: deeper look at differentials

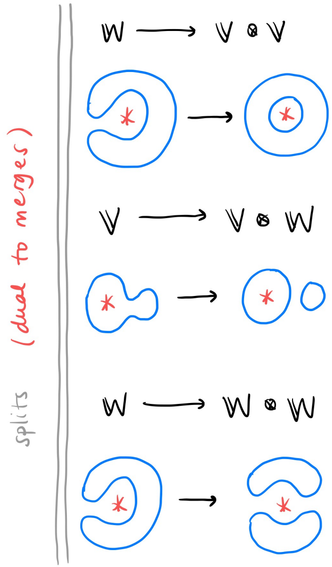
There are now 6 different scenarios along edges of the cube of resolutions:



# Annular Khovanov Homology: deeper look at differentials




ex. Compute $d_{AKh}$ for the cases $V \otimes W \longrightarrow V$ and $W \otimes W \longrightarrow W$.

## History + Motivation

[Jones 1984] Jones polynomial

[Kauffman 1987] Kauffman bracket

$$\begin{cases} \langle \emptyset \rangle = 1 \\ \langle \bigcirc \sqcup L \rangle = (q + q^{-1}) \langle L \rangle \\ \langle \times \rangle = \langle )( \rangle - q \langle \asymp \rangle \end{cases}$$

+ some q-shift + normalization gives Jones polynomial

[Khovanov 1999] Khovanov Homology

$$\begin{cases} [\![ \emptyset ]\!] = 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \\ [\![ \bigcirc \sqcup L ]\!] = V \otimes [\![ L ]\!] \\ [\![ \times ]\!] = \mathcal{F}\left( 0 \longrightarrow [\![ )( ]\!] \xrightarrow{d_{Kh}} [\![ \asymp ]\!]\{1\} \longrightarrow 0 \right) \end{cases}$$

$\oplus$ along $gr_h$

$gr_q$ shift

$\overbrace{\text{[Asaeda - Przytycki - Sikora}}^{\text{APS}}$ 2004] Categorification of Kauffman bracket skein module of $F \times [0,1]$ (I-bundles over surfaces)

AKh = APS Homology for F = annulus

## History + Motivation

[Jones 1984] Jones polynomial

[Kauffman 1987] Kauffman bracket

[Khovanov 1999] Khovanov Homology

[Asaeda-Przytycki-Sikora 2004] Categorification of Kauffman bracket skein module of $F \times [0,1]$ ($I$-bundles over surfaces)

AKh = APS Homology for $F$ = annulus

[Queffelec-Rose 2015] Annular $\mathfrak{sl}_N$ link homology

- Many results we'll discuss have analogues in annular $\mathfrak{sl}_N$ link homology.
- If an analogue has not been explored, you should explore it!