A Perspective on Annular Khovanov Homology

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Perspectives on Quantum Link Homology Theories

2021 August 9 - 13 University of Regensburg

Roadmap (assuming perfect weather and travel conditions)

Lect. 1
- Notation, definition, computation
- History and Motivation: categorified skein module

Lect. 2
- Application: Periodic Links
- AKh and Floer theories
  - Spectral sequences
  - Knot detection results

Lect. 3
- Structure of AKh:
  - $\mathfrak{sl}_2(\mathbb{C})$ action
- Annular Khovanov-Lee Homology
- Distilled Numerical Invariants: annular Rasmussen invariants

Lect. 4
- Annular filtrations on other Khovanov Homologies
- Applications to
  - knot concordance
  - transverse knots
  - braids as MCG ($\mathbb{D}^2$, some points)
- Braid detection results

Application: Periodic Links

Example of how AKh is useful even when studying nonannular objects

<u>Question</u> If

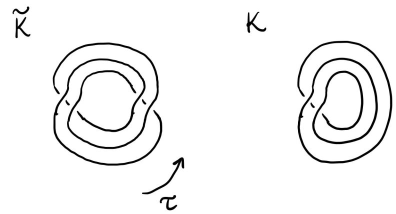
$\widetilde{K} \subset S^3$ periodic knot

$\tau : S^3 \longrightarrow S^3$, $\tau^p = \mathrm{id}$, τ free on $\widetilde{K}$

$K = \widetilde{K}/\tau$ quotient knot

then

is $Kh(\widetilde{K})$ bigger than $Kh(K)$?



<u>Background</u>

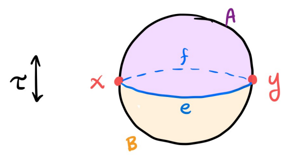
P. A. Smith (~100 years ago)

"Smith-type inequality"

[Borel] $\dim H^*(\widetilde{X}) \geq \dim H^*(X)$

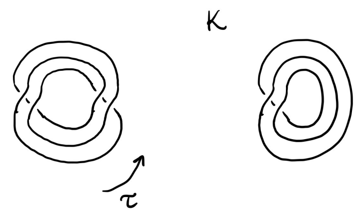
$\widetilde{X}$ = top. space with $X \circlearrowleft \tau$, $\tau^p = \mathrm{id}$

X = fixed-point set



Pf. A spectral sequence $H^*(\widetilde{X}; \mathbb{F}_2) \otimes \Lambda \rightrightarrows H^*(X; \mathbb{F}_2) \otimes \Lambda$ ∎

Application: Periodic Links



Is $Kh(\tilde{K})$ bigger than $Kh(K)$?

Background (focus on $p=2$, $\dim = \dim_{\mathbb{F}_2}$ for now)

[Borel] $\dim H^*(\tilde{X}) \geq \dim H^*(X)$

[Seidel-Smith] Lagrangian Floer Homology (M, L_0, L_1) in some cases

eg. symplectic Khovanov homology $Kh_{symp} \cong \delta$-graded Kh, over $\mathbb{F}_2$

Abouzaid-Smith

see also:

[Hendricks] many contexts: HF of branched double covers, HFK of periodic knots

[Boyle] more on Hendricks's spectral sequences

[Lipshitz-Treumann] Hochschild homology of dg bimodules (eg. bordered Floer homology)

[Lidman-Manolescu] HF of regular covers, uses SWF homotopy type

[Large] extends Seidel-Smith to more generality

Application: Periodic Links

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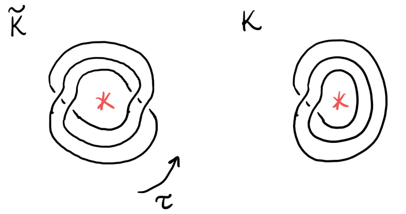


<u>Answer</u> [Stoffregen-Z, Borodzik-Politarczyk-Silvero]

Yes! $\dim_{\mathbb{F}_p} Kh(\widetilde{K}) \geqslant \dim_{\mathbb{F}_p} Kh(K)$ because of Borel-like spectral sequence

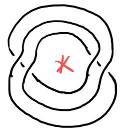
$$Kh(\widetilde{K};\mathbb{F}_p) \otimes \Lambda \Longrightarrow AKh(K;\mathbb{F}_p) \otimes \Lambda$$

and filtration spectral sequence $AKh(K) \Longrightarrow Kh(K)$. ∎

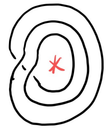
By the Smith Conjecture (true), periodic knots are inherently annular objects.

Application: Periodic Links

$\tilde{K}$



K



[Stoffregen-Z, Borodzik-Politarczyk-Silvero 2019]

$$\dim_{\mathbb{F}_p} Kh(\tilde{K}) \geqslant \dim_{\mathbb{F}_p} Kh(K)$$

Pf. uses Khovanov homotopy type [Lipshitz-Sarkar], [Lawson-Lipshitz-Sarkar].

But a similar result can be proven combinatorially!

<u>theorem</u> [Z 2018] Let $\tilde{K}$ be a 2-periodic knot with quotient knot K.
Then there is a spectral sequence

$$AKh(\tilde{K}, \mathbb{F}_2) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \rightrightarrows AKh(K) \otimes \mathbb{F}_2[\theta, \theta^{-1}].$$

The differentials in this spectral sequence lead to the conjecture of the more general theorem.

Let's see how such Smith-type inequalities are proven by studying this case.

Application: Periodic Links

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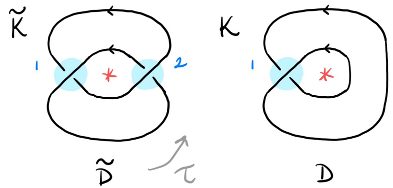

<u>Proof Sketch</u> $^{h}d^2 = 1 - \tau^p = (1-\tau)(1+\tau+\tau^2+\dots+\tau^{p-1})$

① Construct localized Tate bicomplex: underlying: $AKc(\tilde{D}) \otimes \mathbb{F}[\theta, \theta^{-1}]$

$$\begin{tikzcd}
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The bicomplex gives 2 filtration spectral sequences: $^{vh}E_\bullet$, $^{hv}E_\bullet$

② Show that $^{hv}E_\infty \cong AKh(D) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$

★ ⓐ $^{hv}E_3 \cong AKh(D) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$

ⓑ $^{hv}E_3 = {}^{hv}E_\infty$ (no longer differentials)

(this is usually the hardest part)

③ Show that $^{vh}E_1 \cong AKh(\tilde{D}) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$

Finiteness conditions satisfied $\Rightarrow$ $^{vh}E_\infty \cong {}^{hv}E_\infty$

$^{vh}E_\bullet$ is the desired spectral sequence. ■

Application: Periodic Links

<u>theorem</u> [Z 2018] Let $\tilde{K}$ be a 2-periodic knot with quotient knot K. Then

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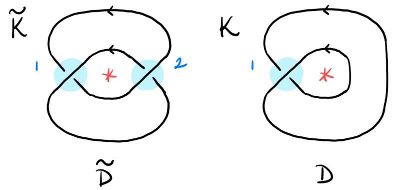
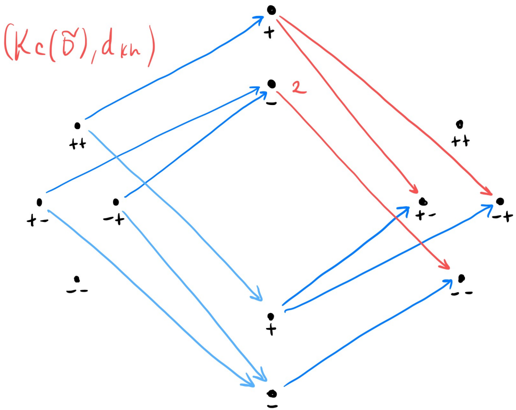


★ is proven case-by-case:

eg.



The bicomplex gives 2 filtration spectral sequences: ${}^{vh}E_\bullet$, ${}^{hv}E_\bullet$

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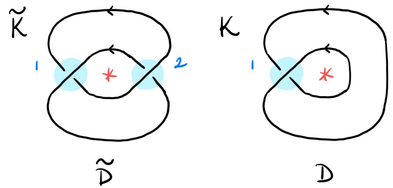

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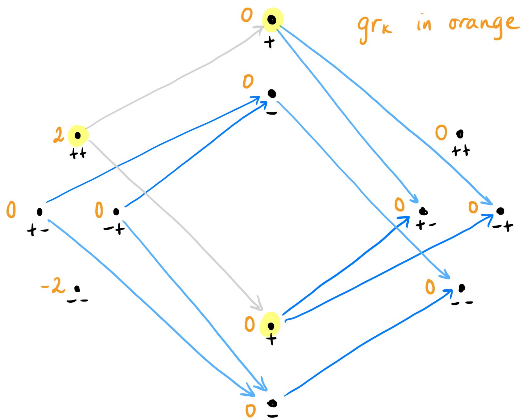

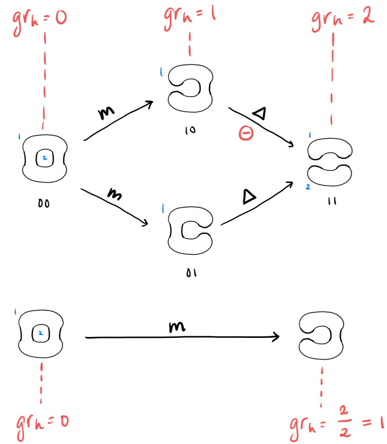

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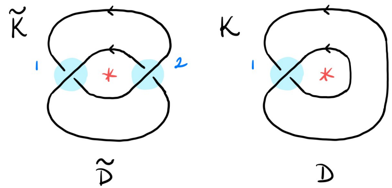

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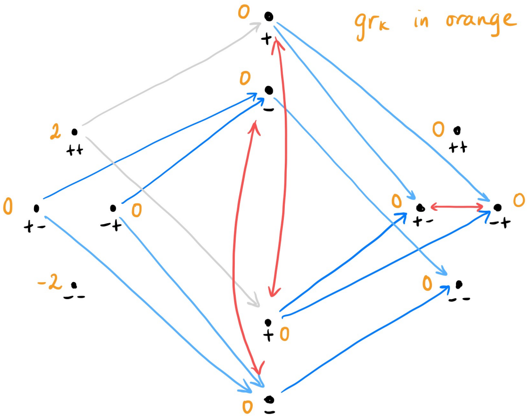

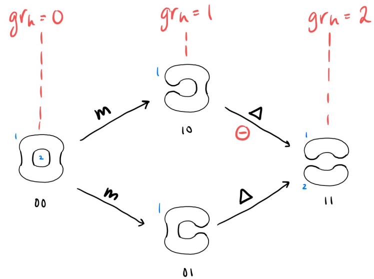

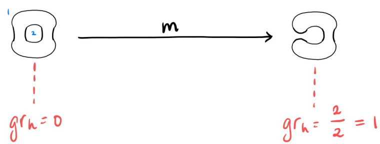

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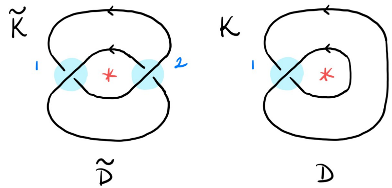

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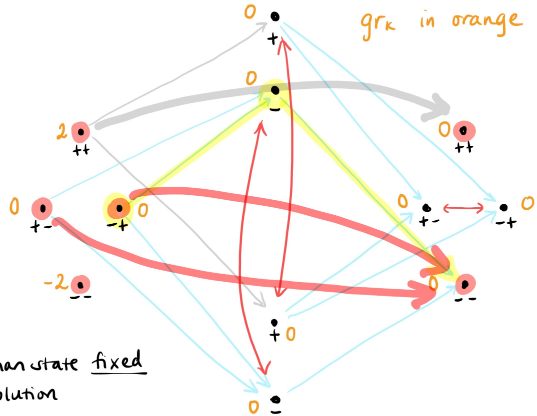
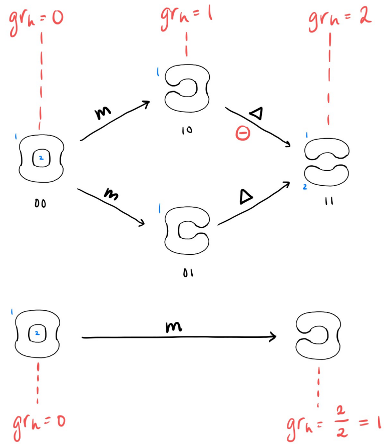


● = Kauffman state <u>fixed</u> by involution



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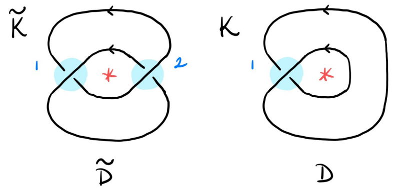

<u>theorem</u> [Z 2018] Let $\tilde{K}$ be a 2-periodic knot with quotient knot K. Then

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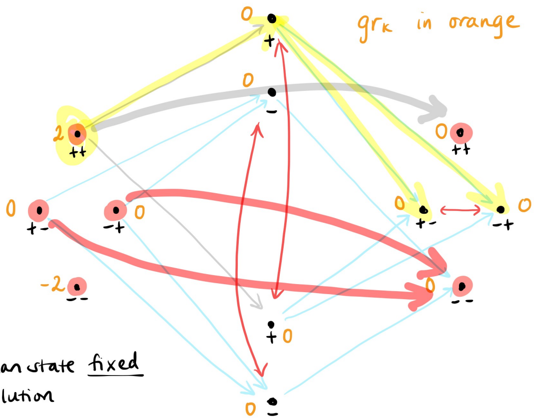
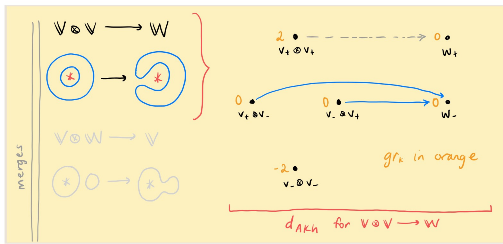


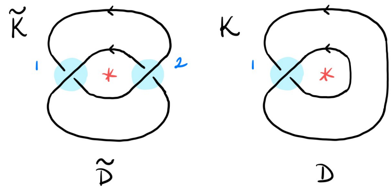
● = Kauffman state <u>fixed</u> by involution

<u>Observation:</u>

$$({}^{hv}E_2, d_2) = (AKc(D), d_{AKh(D)}) \otimes \Lambda\ !$$



Application: Periodic Links



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Proof Sketch

① Construct localized Tate bicomplex: underlying: $AKc(\widetilde{D}) \otimes \mathbb{F}[\theta, \theta^{-1}]$

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By showing $({}^{hv}E_2, d_2) = (AKc(D), d_{AKh}) \otimes \Lambda$

★ ⓐ ${}^{hv}E_3 \cong AKh(D) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$

By grading obstructions.

Recall: AKh is trigraded.

ⓑ ${}^{hv}E_3 = {}^{hv}E_\infty$ (no longer differentials)
(this is usually the hardest part)

③ Show that ${}^{vh}E_1 \cong AKh(\widetilde{D}) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$

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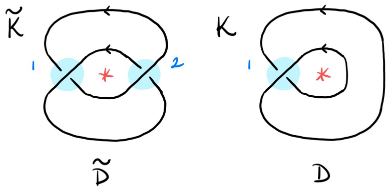
Application: Periodic Links



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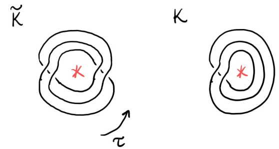
Application: Periodic Links



Question Is $Kh(\tilde K)$ bigger than $Kh(K)$?

Answer [Stoffregen-Z, Borodzik-Politarczyk-Silvero]

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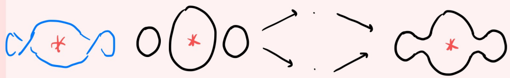
and filtration spectral sequence $AKh(K) \rightrightarrows Kh(K)$.

- The analogue to Step 2b requires the Lipshitz-Sarkar Khovanov homotopy type
- However, the analogue to Step 2a is essentially the same as showing that $({}^{hv}E_2, d_2)$ is still $(AKc(D), d_{AKh})\otimes\Lambda$!

  ↑ ${}^{hv}E_\bullet$ from bicomplex with d_{Kh} vertical differentials

ex. Show that the length 2 induced differential (${}^{hv}d_2$) for the 2-periodic lift of $\mathbb{V}\mathbb{W}\longrightarrow\mathbb{V}$



agree with d_{AKh} for $\mathbb{V}\mathbb{W}\longrightarrow\mathbb{V}$.

(Note that only the $\mathbb{Z}/2$-fixed Kauffman states survive to page ${}^{hv}E_2$.)

Application of Application : Detection Results

T(2,5) is the only knot with its Kh.

<u>theorem</u> [Baldwin-Hu-Sivek 2021] Khovanov homology detects $T(2,5)$.

But whether HFK (knot Floer homology) detects $T(2,5)$ is still open!
(The knot genus is too big to obtain a finite list of knots to check.)

This is the first detection result for Kh (other than for unknot)
that doesn't use detection by HFK!

The proof uses many tools, notably for us

- the spectral sequence $\mathrm{Kh}(\widetilde{K}) \otimes \Lambda \rightrightarrows \mathrm{AKh}(K) \otimes \Lambda$ [SZ, BPS]
- and the $\underbrace{\mathfrak{sl}_2(\mathbb{C})\text{-action on } \mathrm{AKh}(K)}_{\text{Lecture 3 tomorrow!}}$ [Grigsby-Licata-Wehrli] !

Detection Results

Unknot, Unlink

[Submitted on 24 May 2010]

Khovanov homology is an unknot-detector

P. B. Kronheimer, T. S. Mrowka

We prove that a knot is the unknot if and only if its reduced Khovanov cohomology has rank 1. The proof has two steps. We show first that there is a spectral sequence beginning with the reduced Khovanov cohomology and abutting to a knot homology defined using singular instantons. We then show that the latter homology is isomorphic to the instanton Floer homology of the sutured knot complement: an invariant that is already known to detect the unknot.

[Submitted on 4 Apr 2012 (v1), last revised 8 Oct 2012 (this version, v2)]

Khovanov module and the detection of unlinks

Matthew Hedden, Yi Ni

We study a module structure on Khovanov homology, which we show is natural under the Ozsvath-Szabo spectral sequence to the Floer homology of the branched double cover. As an application, we show that this module structure detects trivial links. A key ingredient of our proof is that the H_1/Torsion module structure on Heegaard Floer homology detects S^1xS^2 connected summands.

Detection Results Annular Khovanov Homology (née Sutured Khovanov homology/ sutured annular Khovanov homology)

[Submitted on 10 Dec 2012 (v1), last revised 26 Jul 2013 (this version, v2)]

Categorified invariants and the braid group

John A. Baldwin, J. Elisenda Grigsby

We investigate two "categorified" braid conjugacy class invariants, one coming from Khovanov homology and the other from Heegaard Floer homology. We prove that each yields a solution to the word problem but not the conjugacy problem in the braid group.

Comments: 12 pages, 2 figures. Version 2 has a new, entirely combinatorial, proof that sutured annular Khovanov homology detects the trivial braid, involving a relationship between Plamenevskaya's invariant of transverse links and the Dehornoy order on the braid group

[Submitted on 9 May 2013 (v1), last revised 25 Aug 2014 (this version, v2)]

Sutured Khovanov homology distinguishes braids from other tangles

J. Elisenda Grigsby, Yi Ni

We show that the sutured Khovanov homology of a balanced tangle in the product sutured manifold D x I has rank 1 if and only if the tangle is isotopic to a braid.

Detection Results (more)

[Submitted on 23 Jan 2018 (v1), last revised 13 Apr 2021 (this version, v2)]

Khovanov homology detects the trefoils

John A. Baldwin, Steven Sivek

We prove that Khovanov homology detects the trefoils. Our proof incorporates an array of ideas in Floer homology and contact geometry. It uses open books; the contact invariants we defined in the instanton Floer setting; a bypass exact triangle in sutured instanton homology, proven here; and Kronheimer and Mrowka's spectral sequence relating Khovanov homology with singular instanton knot homology. As a byproduct, we also strengthen a result of Kronheimer and Mrowka on $SU(2)$ representations of the knot group.

[Submitted on 11 Oct 2018]

Khovanov homology detects the Hopf links

John A. Baldwin, Steven Sivek, Yi Xie

We prove that any link in S^3 whose Khovanov homology is the same as that of a Hopf link must be isotopic to that Hopf link. This holds for both reduced and unreduced Khovanov homology, and with coefficients in either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$.

[Submitted on 1 Jul 2019 (v1), last revised 25 Jul 2019 (this version, v2)]

Instanton Floer homology for sutured manifolds with tangles

Yi Xie, Boyu Zhang

We prove an excision theorem for the singular instanton Floer homology that allows the excision surfaces to intersect the singular locus. This is an extension of the non-singular excision theorem by Kronheimer and Mrowka and the genus-zero singular excision theorem by Street. We use the singular excision theorem to define an instanton Floer homology theory for sutured manifolds with tangles. As applications, we prove that the annular Khovanov homology (1) detects the unlink, (2) detects the closure of the trivial braid, and (3) distinguishes braid closures from other links; we also prove that the annular instanton Floer homology detects the Thurston norm of meridional surfaces.

Detection Results (even more)

[Submitted on 6 May 2020]

Khovanov homology detects $T(2,6)$

Gage Martin

We show if L is any link in S^3 whose Khovanov homology is isomorphic to the Khovanov homology of $T(2,6)$ then L is isotopic to $T(2,6)$. We show this for unreduced Khovanov homology with $\mathbb{Z}$ coefficients.

[Submitted on 12 May 2020]

Two detection results of Khovanov homology on links

Zhenkun Li, Yi Xie, Boyu Zhang

We prove that Khovanov homology with Z/2-coefficients detects the link L7n1, and the union of a trefoil and its meridian.

[Submitted on 2 Jul 2020]

Khovanov homology detects the figure-eight knot

John A. Baldwin, Nathan Dowlin, Adam Simon Levine, Tye Lidman, Radmila Sazdanovic

Using Dowlin's spectral sequence from Khovanov homology to knot Floer homology, we prove that reduced Khovanov homology (over $\mathbb{Q}$) detects the figure-eight knot.

[Submitted on 3 Nov 2020]

Knot Floer homology, link Floer homology and link detection

Fraser Binns, Gage Martin

We give new link detection results for knot and link Floer homology inspired by recent work on Khovanov homology. We show that knot Floer homology detects $T(2,4)$, $T(2,6)$, $T(3,3)$, $L7n1$, and the link $T(2,2n)$ with the orientation of one component reversed. We show link Floer homology detects $T(2,2n)$ and $T(n,n)$, for all n. Additionally we identify infinitely many pairs of links such that both links in the pair are each detected by link Floer homology but have the same Khovanov homology and knot Floer homology. Finally, we use some of our knot Floer detection results to give topological applications of annular Khovanov homology.

Application of Application : Detection Results

theorem [Baldwin - Hu - Sivek 2021] Khovanov homology detects $T(2,5)$.

[Submitted on 25 May 2021]

Khovanov homology and the cinquefoil

John A. Baldwin, Ying Hu, Steven Sivek

We prove that Khovanov homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$ detects the $(2,5)$ torus knot. Our proof makes use of a wide range of deep tools in Floer homology, Khovanov homology, and Khovanov homotopy. We combine these tools with classical results on the dynamics of surface homeomorphisms to reduce the detection question to a problem about mutually braided unknots, which we then solve with computer assistance.

But whether HFK (knot Floer homology) detects $T(2,5)$ is still open!
(The knot genus is too big to obtain a finite list of knots to check.)

This is the first detection result for Kh (other than for unknot)
that doesn't use detection by HFK!

The proof uses many tools, notably for us

- the spectral sequence $Kh(\widetilde{K}) \otimes \Lambda \rightrightarrows AKh(K) \otimes \Lambda$ [SZ, BPS]
- and the $\underbrace{sl_2(\mathbb{C})\text{-action on } AKh(K)}_{\text{Lecture 3 tomorrow!}}$ [Grigsby - Licata - Wehrli] !

AKh and Floer Theories

example: work of [L. Roberts] and [Grigsby-Wehrli]

Later in A. Lobb's lectures, you'll encounter [Ozsváth-Szabó]'s spectral sequence

$$\widetilde{Kh}(\bar{K}, \mathbb{F}_2) \Rightarrow \widehat{HF}(\Sigma(S^3, K), \mathbb{F}_2)$$

reduced Kh

($\widetilde{Kh}$ is to Kh as $\widetilde{H}^*$ is to H^*)

mirror of K

Heegaard Floer homology

(3-manifold invariant)

branched double cover of S^3 along $K \subset S^3$

[L. Roberts] Generalizes the Ozsváth-Szabó spectral sequence to relate

$$AKh(\bar{L}', \mathbb{F}_2) \quad \text{and} \quad \widehat{HFK}(\Sigma(K) \# (S^1 \times S^2) \# (S^1 \times S^2), \widetilde{B} \# B(0,0))$$

$\bar{L} \sqcup$ 

lift of z-axis $\cup \{\infty\}$

$B(0,0) \subset S^1 \times S^2 \# S^1 \times S^2$ from

AKh and Floer Theories

example: work of [L. Roberts] and [Grigsby-Wehrli]

[Grigsby-Wehrli] Reinterpreted and extended the previous relationships, relating

$$\mathrm{AKh}(\bar{L}) \quad \text{with} \quad \mathrm{SFH}(\Sigma(A \times I, L))$$

[Juhasz]'s sutured Floer homology

(HFK can be described using the SFH framework.)

Questions and clarifications?