A Perspective On

Annular Khovanov Homology

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Perspectives on Quantum Link Homology Theories

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Roadmap (assuming perfect weather and travel conditions)

Lect. 1
- Notation, definition, computation
- History and Motivation: categorified skein module

Lect. 2
- Application: Periodic Links
- AKh and Floer theories
  - Spectral sequences
  - Knot detection results

Lect. 3
- Structure of AKh:
  - $\mathfrak{sl}_2(\mathbb{C})$ action
- Annular Khovanov-Lee Homology
- Distilled Numerical Invariants: annular Rasmussen invariants
- Annular filtrations on other Khovanov Homologies

Lect. 4
- Applications to
  - knot concordance
  - transverse knots
  - braids as MCG ($\mathbb{D}^2$, some points)
- Braid detection results

Structure of AKh : $\mathfrak{sl}_2(\mathbb{C})$ action [Grigsby - A. Licata - Wehrli]

$\mathbb{V} \cong \mathbb{C}^2$

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

- $\mathbb{C}$-basis $\{e, f, h\}$
- Lie bracket
$$[e,f] = h$$
$$[e,h] = -2e$$
$$[f,h] = 2f$$

Defining 2-dim'l representation: $\mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathrm{End}(\mathbb{V})$:

$$h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

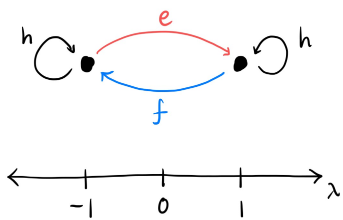
generates maximal torus — raising operator — lowering operator

Any finite-dimensional representation U of $\mathfrak{sl}_2\mathbb{C}$ decomposes (as a $\mathbb{C}$-vector space) into weight-spaces (eigenspaces of h):

$$U = \bigoplus_{\lambda \in \mathbb{Z}} U[\lambda]$$

where $U[\lambda] = \{ v \in U \mid h \cdot v = \lambda v \}$

eg. Defining representation:

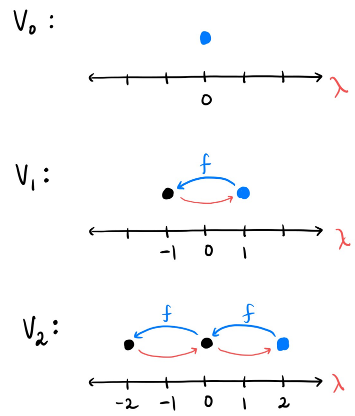


Structure of AKh : $\mathfrak{sl}_2(\mathbb{C})$ action [Grigsby - Licata - Wehrli]

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

Any finite dimensional representation U decomposes into irreducible representations.

The irreducible representations of $\mathfrak{sl}_2\mathbb{C}$ are in bijection with $\{0, 1, 2, 3, \ldots\}$ via their highest weight vector.

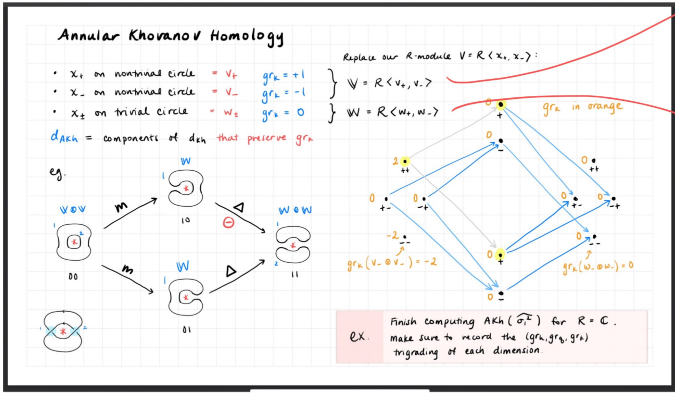


etc.

> [Grigsby - Licata - Wehrli]
> $AKh(L;\mathbb{C})$ has an $\mathfrak{sl}_2$ action.
> i.e. $AKh(L;\mathbb{C})$ is an $\mathfrak{sl}_2$ representation.
> gr_k is the $\mathfrak{sl}_2$ weight-space grading!

Structure of AKh : $sl_2(\mathbb{C})$ action [Grigsby - Licata - Wehrli]

Recall from Lecture 1 :



as sl_2 reps

$$\mathbb{V} = \mathbb{C}\langle v_+, v_- \rangle \cong V_1$$

$gr_k(v_+) = +1$ $\quad gr_q(v_\pm) = \pm 1$

$gr_k(v_-) = -1$

$$\mathbb{W} = \mathbb{C}\langle w_+, w_- \rangle \cong V_0 \oplus V_0$$

$gr_k(w_\pm) = 0 \quad gr_q(w_\pm) = \pm 1$

(dual representation to $\mathbb{V}$)

$$\mathbb{V}^* = \mathbb{C}\langle v_+^*, v_-^* \rangle \cong \mathbb{V} \cong V_1$$

$gr_k(v_+^*) = -1 \quad gr_q(v_+^*) = -1$

$gr_k(v_-^*) = +1 \quad gr_q(v_-^*) = +1$

We need to

1. assign $\mathbb{V}$, $\mathbb{V}^*$, and $\mathbb{W}$ to complete resolutions,
2. define the actions of the operators $e, f,$ and h,
3. and check that the differential respects the sl_2-action on chains.

Structure of AKh : $\mathfrak{sl}_2(\mathbb{C})$ action [Grigsby-Licata-Wehrli]

1. assign $\mathbb{V}$, $\mathbb{V}^*$, and $\mathbb{W}$ to complete resolutions

Define a functor $\mathcal{F} : Cob^3_{/\ell}(A) \longrightarrow gRep(\mathfrak{sl}_2)$.

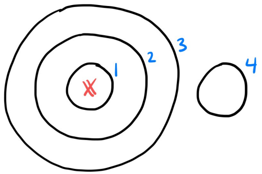
BASES

$$\mathbb{V} = \mathbb{C}\langle \overset{+1}{v_+}, \overset{-1}{v_-} \rangle$$

$$\mathbb{W} = \mathbb{C}\langle \overset{0}{w_+}, \overset{0}{w_-} \rangle$$

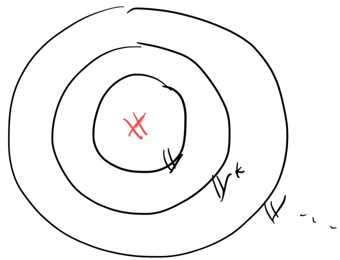
$$\mathbb{V}^* = \mathbb{C}\langle \underset{-1}{\overset{-1}{v_+^*}}, \underset{+1}{\overset{+1}{v_-^*}} \rangle$$

eg.



$\longrightarrow$ assign $\mathbb{V} \otimes \mathbb{V}^* \otimes \mathbb{V} \otimes \mathbb{W}$

- circle 4 is trivial $\longrightarrow$ assign $\mathbb{W}$
- circles 1, 2, 3 are nontrivial
  - circles 1 and 3 enclose an even # of nontrivial circles $\longrightarrow$ assign $\mathbb{V}$
  - circle 2 encloses an odd # of nontrivial circles $\longrightarrow$ assign $\mathbb{V}^*$

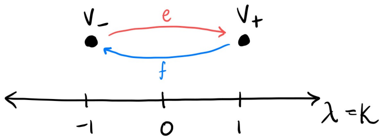


Structure of AKh : $\mathfrak{sl}_2(\mathbb{C})$ action [Grigsby - Licata - Wehrli]

2. define the actions of the operators $e, f,$ and h

(We're just choosing a good basis. Since $\mathbb{V}, \mathbb{V}^*,$ and $\mathbb{W}$ are already $\mathfrak{sl}_2$ reps, the bracket relations are already satisfied.)

eg. $e, f,$ and h on $\mathbb{V} = \mathbb{C}\langle v_+, v_- \rangle$



$$e \cdot v_+ = 0, \quad f \cdot v_- = 0$$

$$h \cdot v_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \cdot v_+$$

$$e \cdot v_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_+$$

$$h \cdot v_- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1 \cdot v_-$$

$$f \cdot v_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_-$$

BASES

$$\mathbb{V} = \mathbb{C}\langle \overset{+1}{v_+}, \overset{-1}{v_-} \rangle$$

$$\mathbb{W} = \mathbb{C}\langle \overset{0}{w_+}, \overset{0}{w_-} \rangle$$

$$\mathbb{V}^* = \mathbb{C}\langle \underset{-1}{\overset{-1}{v_+^*}}, \underset{+1}{\overset{+1}{v_-^*}} \rangle$$

Structure of AKh : $\mathfrak{sl}_2(\mathbb{C})$ action [Grigsby - Licata - Wehrli]

2. define the actions of the operators $e, f,$ and h

(We're just choosing a good basis. Since $\mathbb{V}$, $\mathbb{V}^*$, and $\mathbb{W}$ are already $\mathfrak{sl}_2$ reps, the bracket relations are already satisfied.)

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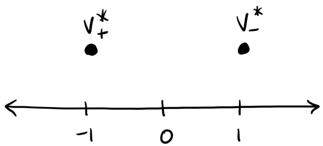
$$\mathbb{V}^* = \mathbb{C}\langle \underset{-1}{\overset{-1}{v_+^*}}, \underset{+1}{\overset{+1}{v_-^*}} \rangle$$

eg. $e, f,$ and h on $\mathbb{V}^* = \mathbb{C}\langle v_+^*, v_-^* \rangle$



$$e \cdot v_-^* = 0, \quad f \cdot v_+^* = 0$$

$$h \cdot v_+^* = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1 \cdot v_+^*$$

$$e \cdot v_+^* = -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -v_-^*$$

$$h \cdot v_-^* = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 1 \cdot v_-^*$$

$$f \cdot v_+^* = -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -v_+^*$$

Structure of AKh : $\mathfrak{sl}_2(\mathbb{C})$ action [Grigsby - Licata - Wehrli]

3. check that the differential respects the $\mathfrak{sl}_2$-action on chains.

This needs to be checked case-by-case.

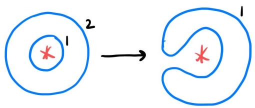
BASES

$\mathbb{V} = \mathbb{C}\langle \overset{+1}{v_+}, \overset{-1}{v_-} \rangle$

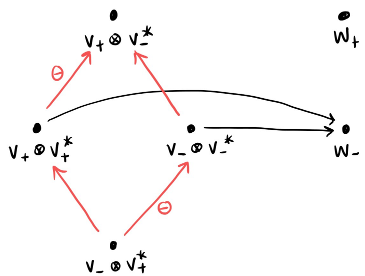
$\mathbb{W} = \mathbb{C}\langle \overset{0}{w_+}, \overset{0}{w_-} \rangle$

$\mathbb{V}^* = \mathbb{C}\langle \overset{-1}{\underset{-1}{v_+^*}}, \overset{+1}{\underset{+1}{v_-^*}} \rangle$

eg. $\mathbb{V} \otimes \mathbb{V}^* \longrightarrow \mathbb{W}$



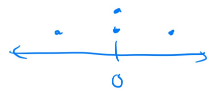
Need:
$$e \circ d_{AKh} = d_{AKh} \circ e$$
$$f \circ d_{AKh} = d_{AKh} \circ f$$
$$h \circ d_{AKh} = d_{AKh} \circ h$$



Note: $\mathfrak{sl}_2 \curvearrowright \mathbb{V} \otimes \mathbb{V}^*$ by a "Leibniz formula":

eg. $e \cdot (v_- \otimes v_+^*)$
$$= (e \cdot v_-) \otimes v_+^* + v_- \otimes (e \cdot v_+^*)$$
$$= v_+ \otimes v_+^* + v_- \otimes (-v_-^*)$$
$$= v_+ \otimes v_+^* - v_- \otimes v_-^*$$

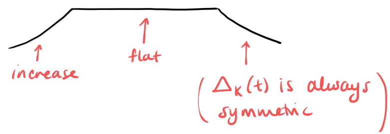
$\mathbb{V} \otimes \mathbb{V}^*$:



ex.	Confirm that $\mathbb{V} \otimes \mathbb{W} \xrightarrow{d_{AKh}} \mathbb{V}$ also commutes with the $\mathfrak{sl}_2$ action.

Remarks

a "trapezoid":



① Trapezoidality

Conjecture [Fox 1962]

The coefficients of the Alexander polynomial $\Delta_K(t)$ of an alternating knot K are <u>trapezoidal</u>.

$\chi(HFK)$

∃ a spectral sequence relating them

Recall that $AKh \rightsquigarrow \widehat{HFK}$!

Corollary to $\mathfrak{sl}_2 \curvearrowright AKh$ [Grigsby - Licata - Wehrli]

The dimensions of $AKh^{*,*,k}(L)$ are trapezoidal.

Pf. • The gr_k support of $AKh(L)$ lies within $\underbrace{(\text{wrapping number}}_{} \text{ of } D(L)) + 2\mathbb{Z}$

minimum # times $D(L)$ crosses an arc γ connecting the 2 boundary components of A

note that $wrap(D(L)) + 2\mathbb{Z}$ is invariant under choice of $D(L)$.

• $\mathfrak{sl}_2$ irreps are symmetric in weight spaces, centered at 0. ■

Open	Investigate whether $AKh \rightsquigarrow \widehat{HFK}$ tells us anything about Fox's trapezoidality conjecture. (Use $AKh_{odd} \circlearrowleft \mathfrak{gl}(1\|1)$ [Grigsby-Wehrli]?)

Remarks

② [Akhmechet - Krushkal - Willis] have described an action of e, f, h on the annular Khovanov stable homotopy type.

③ There's actually even more structure. [Grigsby - Licata - Wehrli] "AKh and knotted Schur - Weyl representations"

- $\mathfrak{sl}_2(\wedge)$ = exterior current algebra of $\mathfrak{sl}_2$ acts on $AKh(L)$

- When L = m-framed n-cable of $K_0 \subset S^3$, inside $A \times I$ (imagine $K = U$, L = torus knot)
  $S_n \curvearrowright AKh(L)$. The action commutes with the $\mathfrak{sl}_2(\wedge)$ action, preserves $(gr_h, gr_q - gr_k, gr_k)$ trigrading, and behaves well under cobordisms.

- So if $K \subset A \times I$, we may define n-colored AKh:

$$\underbrace{AKh_n(K)}_{\text{framed knot}} := AKh(K^n)^{S_n} \subset AKh(\underbrace{K^n}_{\text{n cable}})$$

(Pause before we move on to annular Khovanov-Lee homology)

Annular Khovanov-Lee Homology

also [Grigsby-Licata-Wehrli]!

(filtered version over $\mathbb{F}[X]/(X^2-1)$,
as opposed to graded over $\mathbb{F}[X,t]/(X^2-t)$)

Recall Khovanov-Lee Homology

- chains $= Kc(D)$
- differential $= d_{Lee}$

$$= d_{Kh} + \Phi_{Lee}$$

$$= d_{AKh} + d_{-} + \Phi_{Lee}$$

$$= d_{0,0} + d_{0,-2} + d_{4,0} + d_{4,2}$$

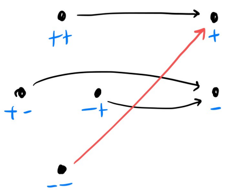
(gr_q, gr_k) bi-degrees of differentials

$$= d_{0,0} + d_{0,4} + d_{4,4} + d_{4,0}$$

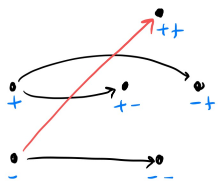
$(gr_q - 2\cdot gr_k, gr_q)$ bi-degrees

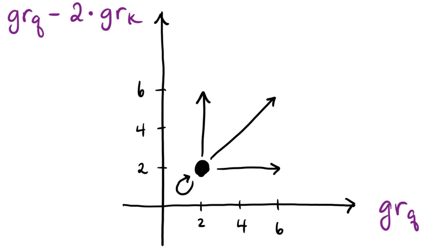
(The following grading conventions follow those in Rasmussen's s-invariant paper.)

merge



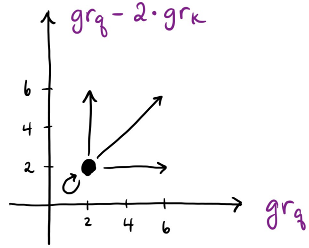
split





Annular Khovanov-Lee Homology

[Grigsby-Licata-Wehrli]



What can we do with this?

⓪ Study its structure

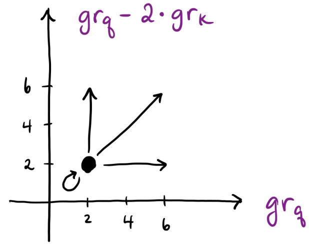
① Use the gradings to extract Rasmussen-s-like invariants d_t

② Use d_t to study
- annular knots (obviously)
- transverse knots

  $= \text{braid closures} \Big/ \underbrace{\text{isotopy}}_{(\text{Markov } 1)} + \text{Markov } 2$
- braid conjugacy classes

details tomorrow

Annular Khovanov-Lee Homology [Grigsby-Licata-Wehrli]



- This is just Lee Homology, but with an extra filtration grading $gr_q - 2\cdot gr_K$.

  But wait! Looking at the differentials, $gr_q - t\cdot gr_K$ $\forall\ t \in [0,2]$ is also a filtration grading!

  <u>defn</u> The annular $\underbrace{\text{concordance}}_{\substack{\text{invariant under annular cntd. cobordisms} \\ F: K_0 \to K_1 \text{ where } \chi(F)=0.}}$ invariant d_t = Rasmussen's s-invt except where you replace gr_q with $gr_t := gr_q - t\cdot gr_K$.

- And functoriality / behavior under cobordisms is also understood from Kh-Lee homology (eg. use Bar-Natan's cobordism category)

ex. if you are familiar with $Cob^3/_\ell$	Compute the gr_t filtration degree of the following cobordisms: ⓐ annular birth (birth of trivial circle) ⓑ nonannular birth (birth of nontrivial circle) ⓒ saddle (saddles are generically annular)

Annular Khovanov-Lee Homology

- One can view this as a strategy for constructing more annular invariants. For example:

[Truong-Z] For an annular link $L \subset A \times I$, the Sarkar-Seed-Szabó perturbation of Szabó's geometric spectral sequence admits an annular filtration.

$$d_{sss} = d_{Sz} + h_{BN}$$

generalizes Bar-Natan's perturbation

$$= \sum_{i=1} d_i + \sum_{i=1} h_i$$

$d_1 = d_{Kh}$

h_1 is like Φ_{Lee}, but works over $\mathbb{F}_2$.

We define analogous invariants $s_{r,t}$ (cf. d_t) and obtain similar applications.

exploratory ex.	Pick your favorite link homology theory. Check if the presence of the annular axis gives an extra grading on generators. Determine if that grading, or a (non-trivial) linear combination of that grading and an existing one gives a filtration grading.

[Akhmechet] has defined $\underbrace{U(1) \times U(1)\text{-equivariant}}_{\text{c.f. [Khovanov-Robert]}}$ annular Khovanov homology.

Annular Khovanov-Lee Homology : Background for topological applications

Why annular invariants?

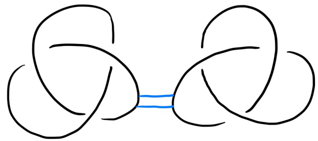
Knot Concordance

$$\mathcal{C} = \mathcal{C}_{smooth} = \{\text{Knots} \subset S^3\} \Big/ \text{concordance} \left(\begin{array}{l} F: K_0 \to K_1, \quad F \subset S^3 \times I \\ F \text{ "looks" like } \end{array} \right)$$

is an abelian group:

- identity $=$ $[U]$
  $= \{$"slice" knots$\}$
  K that bound
  $D^2 \hookrightarrow B^4$
- addition $=$ $\#$
- inverse $=$ mirror

eg.



$K \# m(K)$

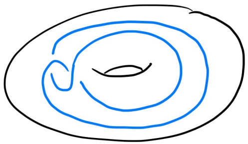
ex.	Try to "see" the D^2 properly embedded in B^4 with $\partial D^2 = K \# m(K)$.

Annular Khovanov-Lee Homology : Background for topological applications

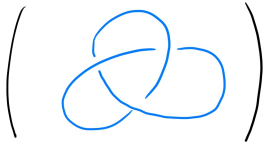
Why annular invariants?

Knot Concordance : Satellite operations

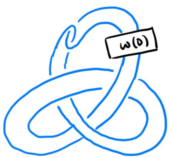
P = pattern $\subset A \times I \approx$ solid torus $\qquad K \subset S^3$ "companion" knot



$\hookrightarrow$ nbhd



Satellite knot $P(K)$:



Fact. If annular knots P_1, P_2 are annularly concordant, then P_1 "=" P_2 as morphisms $P_i : \mathcal{C} \to \mathcal{C}$.

$[K] \longmapsto [P_i(K)]$

Open	Is the converse true or false? i.e. If $P_1 = P_2 : \mathcal{C} \to \mathcal{C}$, are P_1 and P_2 annularly concordant?

Questions and clarifications?