(Towards) Bridge multisections for symplectic surfaces in Weinstein 4-manifolds

Melissa Zhang (UC Davis)

in collaboration with **Román Aranda, Patricia Cahn, Marion Campisi, James Hughes, Daniela Cortes Rodriguez,** and **Agniva Roy**
a.k.a. [ACC$^+$]

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Information

Wednesday, January 3, 2024
3:00 PM - 3:30 PM
Room 023 (Exhibition Level, The Moscone Center)

Abstract

Islambouli and Starkston describe two algorithms for encoding Weinstein 4-manifolds using multisections with divides. In this talk, we report on progress toward encoding embedded symplectic surfaces in their construction, by adapting Meier and Zupan's bridge trisection techniques to the symplectic setting. This is joint work with Román Aranda, Patricia Cahn, Agniva Roy, James Hughes, Marion Campisi, and Daniela Cortes Rodriguez.

Describing Smooth 3- and 4-Manifolds

We will start by discussing the **bolded** terms below.

To capture the smooth structure of a ...	*we have a diagrammatic tool:*
3-manifold Y^3	**Heegaard diagram** $\mathcal{H}$ [Morse, Smale, Singer]
1-manifold L properly embedded in Y^3	pairs of basepoints **z** in the Heegaard diagram $\mathcal{H}$ [Ozsváth-Szabo, Rasmussen]
4-manifold X^4	**trisection diagram** $\mathcal{T}$ [Gay-Kirby]
2-manifold F properly embedded in X^4	**bridge trisections** [Meier-Zupan]

Note that all *smooth* 3- and 4-manifolds have handle decompositions.

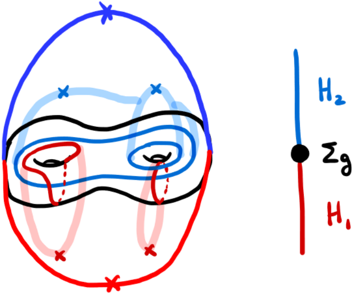
Heegaard Splittings and Diagrams

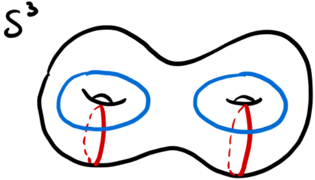
Let Y^3 be a closed, connected, oriented 3-manifold.
A **Heegaard splitting** for Y^3 is a decomposition $Y^3 = H_1 \cup_{\Sigma_g} H_2$ where

- $\Sigma_g = H_1 \cap H_2$ is a genus g orientable surface
- $H_i \cong \natural_g S^1 \times D^2$

A compatible **Heegaard diagram** for the Heegaard splitting (Σ_g, H_1, H_2) is a collection $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta})$ where

- $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$ is a collection of red attaching circles for H_1 and
- $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_g\}$ is a collection of blue attaching circles for H_2





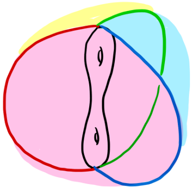
Trisections [Gay-Kirby]

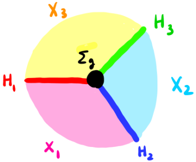
Let X^4 be a closed, connected, oriented 4-manifold.

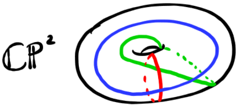
Simplified definition for the purposes of this talk:

A **trisection** of X^4 is a decomposition $X^4 = X_1 \cup X_2 \cup X_3$ where

- $X_i \cong \natural_g S^1 \times D^3$ ($X_i \cong \natural_{k_i} S^1 \times D^3$)
- $H_i := X_i \cap X_{i+1} \cong \natural_g S^1 \times D^2$ for $i = 1, \ldots, n-1$
- $\Sigma_g := X_1 \cap \cdots \cap X_n = \partial H_i$ for all i







Generalizations [Islambouli-Naylor]

- bisection (every smooth, compact, connected 2-handlebody with connected boundary has one)
- multisection (The term *multisection* is overloaded, but the meaning should be clear from context.)

Bridge Trisections [Meier-Zupan]

Theorem [Meier-Zupan]

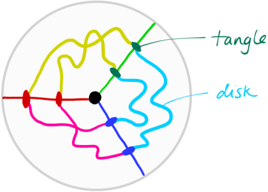
Any smoothly embedded surface $F \subset X^4$ can be isotoped to be in *bridge position* with respect to a given trisection of X^4

For the sake of this talk, let us focus on $X^4 = S^4$.

Bridge Trisections [Meier-Zupan]

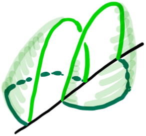
A (balanced) $(b;c)$ **bridge trisection** $\mathcal{T}$ of a knotted surface $F \subset S^4$ is a decomposition $(S^4, F) = (X_1, D_1) \cup (X_2, D_2) \cup (X_3, D_3)$ where

- $S^4 = X_1 \cup X_2 \cup X_3$ is the standard genus 0 trisection of S^4
- (X_i, D_i) is a trivial c-disk system
- $(B_{ij}, \alpha_{ij}) := (X_i, D_i) \cap (X_j, D_j)$ is a b-strand trivial tangle

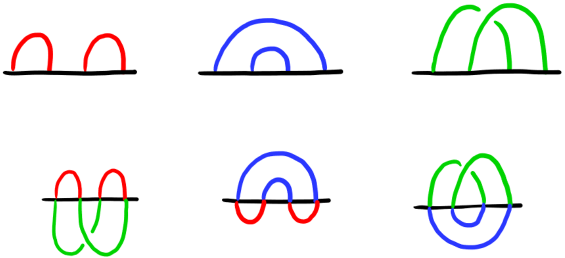


"Trivial?"

- **trivial tangle**: one can simultaneously push every component to the boundary $S^2 = \partial B^3$ (the bridge sphere)
- **trivial disk system**: one can simultaneously push every disk to the boundary $S^3 = \partial B^4$



A **tri-plane diagram** P is a triple of b-strand trivial tangle diagrams (P_{12}, P_{23}, P_{31}) where $P_{ij} \cup \overline{P_{ki}}$ is a diagram for an unlink L_i. Here $\overline{P_{ki}}$ is the the mirror of the tangle diagram P_{ki}.



Describing Contact 3-Manifolds and Symplectic 4-Manifolds

We now add contact / symplectic structure.

To capture the structure of a ...	*we have a diagrammatic tool:*
contact 3-manifold (Y^3, ξ)	**contact Heegaard diagram** $(\mathcal{H}, D)$ [Giroux, Torisu]
transverse knot in (Y^3, ξ)	pairs of basepoints $\mathbf{z}$ in the Heegaard diagram $(\mathcal{H}, D)$
symplectic 4-manifold X^4	Weinstein trisections [Lambert-Cole–Meier-Starkston]
Weinstein domain (W^4, ω)	**multisection with divides** [Islambouli-Starkston]
ascending symplectic surface $F \hookrightarrow (W^4, \omega, \rho, V_\rho)$	**bridge multisections with divides** [ACC$^+$, in progress]

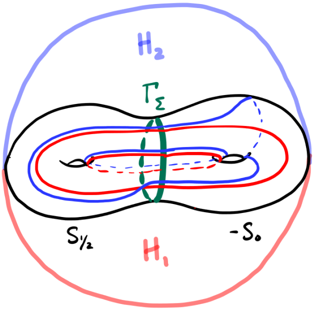
Contact Heegaard Splittings [Giroux, Torisu]

A **contact Heegaard splitting** of (Y^3, ξ) is a Heegaard splitting $Y^3 = H_1 \cup_{\Sigma_g} H_2$ such that

- $\Sigma_g := H_1 \cap H_2$ is a convex surface with dividing set Γ_Σ
- H_i is contactomorphic to a standard neighborhood of its Legendrian spine (i.e. core graph) L_i.

In this case, we (nontrivially!) have:

- H_1 and H_2 are actually two halves of an open-book decomposition supporting (Y^3, ξ)
- For each H_i, there is a set of compression disks cutting H_i into a ball such that the boundary of each compression disk intersects Γ_Σ in exactly two points. In other words, the red α_i and blue β_i can be chosen to each intersect Γ_Σ at exactly two points.



Transverse links

Theorem [Pavelescu]

Any transverse link in a contact 3-manifold (Y^3, ξ) with compatible open book (S, ϕ) can be isotoped to be transverse to the pages of the open book.

In other words, the transverse link can be isotoped through transverse links until it is braided with respect to the open book.

Multisections with Divides for Weinstein Domains [Islambouli-Starkston]

Let (W, ω) be a symplectic filling for its contact boundary $(\partial W, \xi)$.
A **multisection with divides** for (W, ω) is a decomposition $W = W_1 \cup \cdots \cup W_n$ where

- $W_i \cong \natural_{k_i} S^1 \times D^2$
- $H_{i+1} := W_i \cap W_{i+1} \cong \natural_g S^2 \times D^2$ for $i = 1, \ldots, n-1$
- $\Sigma_g := W_1 \cap \cdots \cap W_n = \partial H_i$ for all i
- each $(W_i, \omega|_{W_i})$ is a symplectic filling of $(\partial W_i, \xi_i)$
  $\cong \#_{k_i} S^1 \times S^2$; filling is essentially unique, and Weinstein
- $H_i \cup H_{i+1}$ is a contact Heegaard splitting of $(\partial W_i, \xi_i)$
- $H_1 \cup H_{n+1}$ is a contact Heegaard splitting of $(\partial W, \xi)$

Bridge Multisections Analogue for Ascending Symplectic Surfaces

We [ACC+] are working on adapting Meier-Zupan's strategy to construct **bridge multisections with divides** for certain nicely embedded symplectic surfaces in some of Islambouli-Starkston's multisected Weinstein domains.

Ascending Symplectic Surfaces [Hayden]

Let $(W^4, \omega, \rho, V_\rho)$ be a Weinstein domain (ρ is the Morse function). A smoothly embedded oriented surface $(F, \partial F) \subset (W, \partial W)$ is **ascending** if

- F contains no critical points of $\rho : W \to \mathbb{R}$
- $\rho|_F$ is a Morse function
- each non-critical level set $\rho|_F^{-1}(c) \subset F$ is positively transverse to the contact structure on $\rho^{-1}(c) \subset W$.

[Boileau-Orevkov] studied the B^4 case.

Ascending Symplectic Surfaces [Hayden]

If the ascending surface only has positive critical points, then the transverse link at every critical level is **quasipositive**.

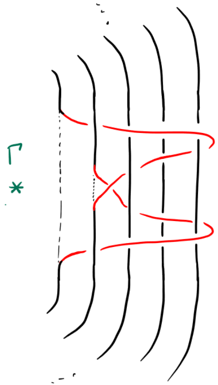
We can then view each surface as a movie of quasipositive braid closures featuring

- transverse isotopy
- quasipositive band moves
- symplectic births.

The Case of $(W^4, \omega) = (B^4, \omega_{std})$

Consider the standard (B^4, ω_{std}) with boundary (S^3, ξ_{rot}).

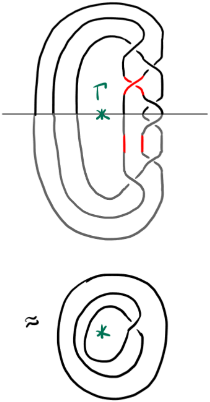
- page of open book $\cong D^2$
- transverse links = braid closures
- dividing set = braid axis



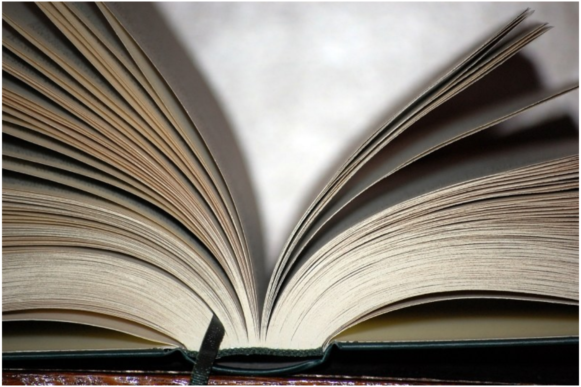
[Boileau-Orevkov] The quasipositive braided boundary **determines** a unique symplectic filling.

The Case of $(W^4, \omega) = (B^4, \omega_{std})$

1. Arrange the quasipositive band moves in different sectors of the multisection to ensure transverse unlinks (consisting of maximal self-linking unknots) on the boundary of each sector.
   - For example, if braids β_1 and β_2 differ at only one positive crossing, then $\beta_1 \cup \overline{\beta_2}$ is indeed isotopic to a transverse unlink with max sl components.
2. Such an unlink bounds a unique collection of trivial symplectic disks in B^4.
3. We can also reduce the number of sectors (to a bisection), by positively stabilizing (a lot) and arranging the quasipositive bands to interact with disjoint pairs of strands.



There's a lot more to do in the general case.



Thank you for listening!