Lecture 2

Make some choices based on student knowledge.

HW01 is up: write as much or as little as you like — submit sth.

last problem: 150-level proof (ring homs)

No Office Hours today:
I have a meeting; may need to change
currently set hours to other day of week

Let R be a ring.

<u>defn.</u> A left R-module is an (additive) abelian group M

equipped w/ scalar multiplication by R: $(r,m) \mapsto rm$ or $r \cdot m$
(ie an action by R)

that is ① distributive in all the reasonable ways

$r(m+m') = rm + rm'$ $\qquad$ $(r+r')m = rm + r'm$

② associative (as in any action) $(rr')m = r(r'm)$

③ $1m = m$

<u>Notation</u> If R acts on M on the left, $M = {}_RM$ is a left R-module.

<u>defn</u> Similar for Right R module $M = M_R$.

$(m+m')r = mr + m'r$ $\qquad$ etc.

<u>However</u> Sometimes people still write "coefficients" on the left. So then we might want to write

$r \cdot m = mr$, in which case

$(rr') \cdot m = mrr' = r' \cdot (r \cdot m)$

doesn't matter if R is comm.

My suggestion is to stick with a notation that makes the most sense at first until it's not confusing anymore.

You've studied modules over comm. rings. Important examples

if R is commutative, enough to write "R-mod" as the cat.

① vectorspaces over fields

② R as module over itself

③ let $T: V \longrightarrow V$ be a linear trans. where V is fin. diml vectorspace over field k.

Then V becomes a $k[x]$-module:

$$k[x] \times V \longrightarrow V$$

$$\left(\sum_{i=0}^{m} c_i x^i, v\right) \longmapsto \sum c_i T^i(v)$$

Book notation: the $k[x]$-mod V is denoted V^T

eg. $A \in Mat_n(\mathbb{R})$ $\quad f(x) = \sum_{i=0}^{m} c_i x^i$

$$f(x)\, w = \sum c_i A^i w$$

Can generalize to $\varphi: M \to M$ $\quad \in End_{k\text{-mod}}(M)$

$$f(x) m = \sum c_i \varphi^i(m) \qquad f(x) \in k[x].$$

↳ skip today unless reviewing repns.

<u>defn</u> A <u>representation</u> of a ring R is a ring hom

$\sigma: R \longrightarrow End(M)$ where M is an abelian group

equiv: $R \times M \to M$

<u>recall</u>: we said $End(A)$ is a ring

ie elements of a ring can be studied as ~~linear~~ operators

Equivalently, M is an R-module. (faithful action)

↳ a repn left R mod is <u>faithful</u> if $\forall r \in R$, $rm = 0\ \forall m \in M \Rightarrow r = 0$. (trivial annihilators)

Examples of modules over noncommutative

① I = left ideal in R : $RI \subset I$.

$\Rightarrow$ I is also a left R mod. can write cat as ${}_R\mathrm{Mod}$ to be clear

(similarly for right ideals + right modules)

$\Rightarrow \{\text{left mods of } R\} \neq \{\text{right mods of } R\}$

eg the 2×2 matrix example we saw

② Say $i : S \hookrightarrow R$. (subring)

Then $S \curvearrowright R$ on both the left and right (different actions!)

eg. $k \hookrightarrow k[X]$. As usually defined, $k[X]$ is a left k-mod.

(because of the associative condition:

$(c_1 c_2)X = c_1(c_2 X)$)

We can of course make a weird right-mod defn:

③ G = group, k = comm ring. Recall kG = group ring.

Let A be a kG-mod. (written multiplicatively w/o special symbols)

Define a new action $G \curvearrowright A$ by

$$g * a = g^{-1} a$$

$$\Rightarrow \left(\sum_{g \in G} c_g g\right) * a = \sum_g c_g g^{-1} a$$

$\Rightarrow$ because $(g_1 g_2)^{-1} = g_2^{-1} \circ g_1^{-1}$,

we now can view A as a right kG-mod.

Morphisms in the cat. of left R-modules:

Let $M = {}_RM$, $N = {}_RN$ be left R-mods.

<u>defn</u> $f: M \to N$ is an <u>R-homomorphism</u> (or R-map)

if ① $f(m+m') = f(m) + f(m')$ group hom

② $f(rm) = r \cdot f(m)$ respect scalar mult

(This is the same as for vector spaces, except R might not be commutative.)

For M_R, N_R: ① same ② $f(mr) = f(m)r$

<u>R-isomorphism</u>: when f is also a bijection.

Then f^{-1} is also an R-map. same proof as usual

eg. $M = {}_RM$. Let $r \in Z(R)$ (center)

Then multiplication by r, $\mu_r : M \to M$, $m \mapsto rm$

is a R-map:

$$a \in R,\ m \in M \Rightarrow \mu_r(am) = ram = arm = a\mu_r(m).$$

<u>Rmks</u> As in the comm case, $\underbrace{\mathrm{Hom}_{R\text{-mod}}(M,N)}$

is an abelian group

will write as $\mathrm{Hom}_R(M,N)$

Module homs can be added: (because N has addition)

$f, g \in \mathrm{Hom}_R(M,N)$

$(f+g)m = f(m) + g(m)$

define

in N

<u>aside</u> "Opposite" R^{op} has mult in opp order as R:

$\mu_{op}(r,s) = \mu(s,r) = sr$.

$\Rightarrow$ If M is a left R-mod, it is automatically a right R^{op}-mod.

not hard to check, but keep track of your notation! Prop 6.15

Prop. Let R be viewed as a left module over itself. } slowly

To be super clear, $R \curvearrowright {}_R R$.

Then $\mathrm{End}_R({}_R R) \cong R^{op}$ as rings

$\mathrm{End}_{R\text{-mod}}({}_R R)$

purely because of how we write fn comp; agrees w/ our matrix multiplication too.

Pf. Need to define an isom

$$\varphi: \mathrm{End}_R({}_R R) \longrightarrow R^{op}$$
$$(f: {}_R R \to {}_R R) \longmapsto f(1)$$

<u>Ring hom:</u>
① group hom
② $\varphi(rr') = \varphi(r)\varphi(r')$
③ $\varphi(1) = 1$.

<u>Check:</u>

③ $\varphi(\mathrm{id}) = \mathrm{id}(1) = 1$.

① $\varphi(f+g) = (f+g)(1) = f(1) + g(1)$

② $\varphi(fg) = (f \circ g)(1)$ b/c fn composition

$= f(g(1))$

Let $r = g(1)$.

Then $f(g(1)) = f(r) = f(r \cdot 1) \overset{f \text{ is a } R\text{-mod map!}}{=} r \cdot f(1) = g(1)f(1)$

In R, $\varphi(f)\varphi(g) = f(1)g(1)$. So in R^{op}, $\varphi(f) \cdot \varphi(g) = g(1)f(1)$.

slowly

<u>defn</u> Anti-isomorphism of rings

$\varphi : R \longrightarrow A$ where $\varphi(rs) = \varphi(s)\varphi(r)$

eg. $R \longrightarrow R^{op}$ (ie $R \cong A^{op}$)

<u>Aside</u> In linear algebra, or even in k-mod where k is comm:

$A \mapsto A^T$ is an anti-isomorphism:

$$(AB)^T = B^T A^T$$

but this only works because $Mat_n(k) \cong Mat_n(k)^{op}$.

If R not comm, then this may not hold:

(the elements in the entries don't nec. commute)

In general, $[Mat_n R]^{op} \cong Mat_n(R^{op})$