Lecture 3

Instructor OH: Tuesdays 3:30–4:30 pm.

Some relevant category theory – framework

§6.2, 6.3 – but see class calendar for the topic names

(don't care what edition you have – HW will be typed out for you)

HW02 will be out tonight, or at the latest tomorrow morning

Words that I won't define in class but you've either covered or still make sense when we talk about left-modules over a non-comm ring R: (Analogous to VS)

- submodule, proper submodule
- R-linear combinations
- cyclic submodule generated by $x \in M$

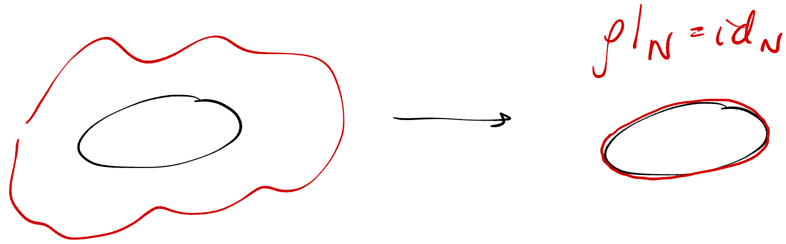
  $\langle x \rangle = Rx$.

  $\rightsquigarrow$ submodule generated by a subset $X \subset M$.
- kernel, image, quotient module, coker
- injection, projection, direct sum

  ? cat theoretically, $\times, \oplus$?
- retraction? $N \subseteq M$ as modules.

  $\rho : M \to N$ is a retraction if $\rho(n) = n \quad \forall\, n \in N$.

  $\rho|_N = id_N$.



  N is a retract of M.
- exact sequence of modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow D$$

(extension of A by C)

Will cover next time if not clear.

R = ring. $S, T \in {}_R\mathrm{Mod}$.

We wish to define $\oplus$, $\times$ in such a way that we can generalize beyond a finite # of factors.

Let $\mathcal{C}$ be a category.

<u>Recall</u> <u>Defn</u> A morphism $f: A \to B$ in $\mathcal{C}$ is an <u>isomorphism</u> if there exists a morphism $g: B \to A$ in $\mathcal{C}$ where

$$\underbrace{gf = 1_A}_{\in\, \mathrm{Mor}_{\mathcal{C}}(A,A)} \quad \text{and} \quad \underbrace{fg = 1_B}_{\in\, \mathrm{Mor}_{\mathcal{C}}(B,B)}.$$

Then g is called the <u>inverse</u> of f.

as opposed to left- or right inverse, which are weaker!

<u>"clear"</u>:
- identity morphisms (aka homs) are always isomorphisms.
- Inverses are unique (why?)

Q. What categories are you already comfy with?

Set, ComRing, Group, ${}_R\mathrm{Mod}$, Mod_R, Top?

(for examples later)

Coproducts: Cat'l.

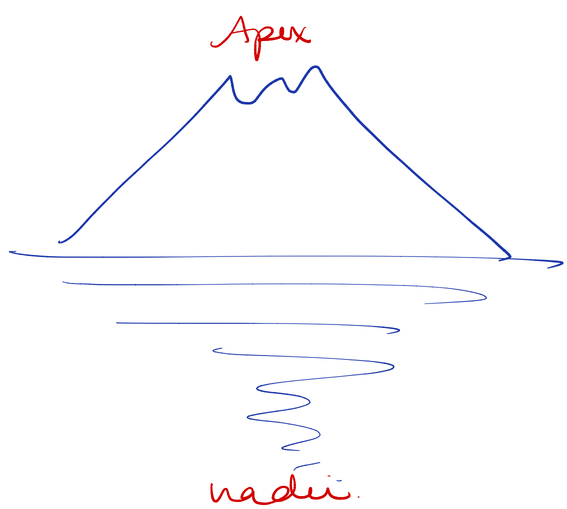
<u>defn.</u> Let $A, B \in Ob(\mathcal{C})$. Their <u>coproduct</u> $A \sqcup B$ ($A \amalg B$) (ie a notion of disjoint union) is an object $C \in Ob(\mathcal{C})$ together w/ <u>injections</u> $\alpha : A \to A \amalg B$

$$\beta : B \longrightarrow A \amalg B$$

such that, $\forall\ X \in Ob(\mathcal{C})$ and morphisms $f : A \to X$, $g : B \to X$ $\exists!$ morphism $\theta : A \amalg B \longrightarrow X$ making the diagram below commute:

Will write $X \in \mathcal{C}$ henceforth

$$\begin{tikzcd}
A \arrow[dr, "\alpha"] \arrow[ddr, bend right, "f"'] & & B \arrow[dl, "\beta"'] \arrow[ddl, bend left, "g"] \\
& A \amalg B \arrow[d, dashed, "\theta"', "\exists!"] & \\
& X &
\end{tikzcd}$$



<u>eg.</u> Coproduct in Sets, Top?

<u>ex.</u> What is coproduct in Groups?

a coproduct (the coproduct)

Prop. If $A, B \in {}_R\mathrm{Mod}$, then $A \amalg B$ $\boxed{\text{exists}}$ in ${}_R\mathrm{Mod}$

and this is the direct sum $C = A \oplus B$ we expect. key word of this proof.

What do we expect? From prev class or analogy...

$C = A \times B$ as set, $\quad A \hookrightarrow C \qquad B \hookrightarrow C$

$\qquad a \mapsto (a, 0) \qquad b \mapsto (b, 0)$

addition by coordinate: $(a,b) + (a',b') = (a+a', b+b')$

scalar mult by coord: $r(a,b) = (ra, rb)$

(can check distributivity easily)

Pf. θ exists:

$$\begin{tikzcd}
A \arrow[dr, "\iota"] \arrow[ddr, bend right, "f"'] & & B \arrow[dl, "j"'] \arrow[ddl, bend left, "g"] \\
& A \oplus B \arrow[d, "\theta"] & \\
& X &
\end{tikzcd}$$

$a \mapsto (a,b) \mapsfrom b$

$(a,b) \mapsto f(a) + f(b)$

θ is unique: If ψ were some other module hom,

$$\psi((a,0)) = f(a)$$

$$\psi((0,b)) = g(b)$$

ψ is a hom

$$\Rightarrow \psi((a,b)) = \psi((a,0)) + \psi((0,b))$$

$$f(a) + g(b) = \theta(a,b).$$

$\square$

Similar proof shows coprods exist in Mod_R.

"<u>Universal Property</u>" a categorical construction (eg. coproduct) X can be defined by a universal property if whenever X exists, it is unique up to unique isom.

This is not a helpful defn — use examples.

(If you want a rigorous defn, use cat theoretic language)

<u>ex.</u> Prove that if a coproduct exists, it is unique up to unique isom.

Solution is in the book... Might also be on HW in some form.

Dual notion: Products (Cat'l)

<u>defn.</u> Let $A, B \in \mathscr{C}$. Then their <u>product</u> $A \sqcap B$ is an object $P \in \mathrm{Ob}(\mathscr{C})$ along with <u>projections</u> $p: P \to A$, $q: P \to B$ such that

$\prod (x-\alpha_i)\ldots$ notation...

I'll just write $A \times B$...

(blah blah, let's draw a picture)

$$\begin{tikzcd}
 & X \arrow[d, "\exists!\,\theta"'] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\
 & P \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd}$$

Commutes.

<u>Prop.</u> In ${}_R\mathrm{Mod}$, products exist and are unique up to unique isom.

pf. HW!

<u>Ex.</u> ① What are $A \sqcup B$ and $A \sqcap B$ in Sets?

② Show these are not isom notions.

<u>Ex.</u> Show $A \oplus B \cong A \times B$ for $A, B \in {}_R\mathrm{Mod}$ (and thus also Mod_R).

pf. in book... maybe also HW.

When the difference matters: infinite indexing sets

<u>defn.</u> $(A_i)_{i\in I}$ I = index set, $A_i \in \mathcal{C}$.

"family of objects" indexed by a set I.

Their <u>coproduct</u> is (the data) $(C, \{\alpha_i : A_i \to C\})$

such that $\exists!\ \theta$ s.t. $\forall i$, this commutes:

$$\begin{tikzcd} A_i \arrow[d, "\alpha_i"] \arrow[dd, bend right, "f_i"'] \\ C \arrow[d, dashed, "\theta"] \\ X \end{tikzcd}$$

<u>defn</u> Similar for product:

$$\begin{tikzcd} X \arrow[d, "\theta"] \arrow[dd, bend left, "f_i"] \\ P \arrow[d, "\alpha_i"'] \\ A_i \end{tikzcd}$$

<u>Cor</u> of prev. prop: $\bigoplus A_i$ and $\prod A_i$ are isom when I is finite (by induction)

Claim: $\bigoplus$ and $\prod$ are not the same in R-Mod! (Same for finite I by induction)

eg. Consider $\mathbb{Z}$-Mod $= Ab$

$$\bigoplus_i A_i = ?$$

$$\prod_i A_i = ?$$

eg. let all $A_i = \mathbb{Z}$. let $I = \mathbb{N}$.

We can only add finitely many things in $\mathbb{Z} \Longrightarrow$

$$\begin{tikzcd}
A_1 \arrow[dr, "i_1"] \arrow[ddr, bend right, "f_1"'] & A_2 \arrow[d, "i_2"] \arrow[dd, bend left, "f_2"] & A_3 \arrow[dl, "i_3"'] \arrow[ddl, bend left, "f_3"] & \cdots \\
 & \bigoplus_{i\in\mathbb{N}} \mathbb{Z} \arrow[d, dashed] & & \\
 & X & &
\end{tikzcd}$$

$$\underbrace{f_1(a_1)+f_2(a_2)+\cdots}_{\text{only makes sense in a } \mathbb{Z}\text{-mod } X \text{ if we have a finite sum.}} = \theta\big((a_1, a_2, \dots)\big)$$

eg. On the other hand:

$$\begin{tikzcd}
 & X \arrow[d, dashed] \arrow[ddl, bend right, "f_1"'] \arrow[dd, bend right, "f_2"'] \arrow[ddr, bend left, "f_3"] & & \\
 & \prod_{i\in\mathbb{N}} \mathbb{Z} \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "p_3"] & & \\
A_1 & A_2 & A_3 & \cdots
\end{tikzcd}$$

no problem ... we can have ∞'ly many nonzero entries.