Lecture 4

HW: Q: finite G for $kG \cong kG^{op}$ as ring?

I'm away Friday! HW will still be posted

We will return to cat. theory concepts as needed.

§6.4 Free + Projective Modules

defn $F \in {}_R\mathrm{Mod}$ is free if $F \cong \bigoplus_{i \in I} R_i$ where $R_i = R\langle b_i \rangle \cong R$.

direct sum!

named basis element

i.e. free b/c there are no relations among the b_i.

note. I could be any indexing set.

eg. $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}$, $\mathbb{Z}^r$; any vector space V over field k.

Q. ∞-dual VS?

We could alternatively define free left R-module in terms of a universal property:

prop. Let F be a free left R-mod w/ basis B.
Then $\forall\, M \in {}_R\mathrm{Mod}$ and set map (function) $\gamma : B \to M$

there exists a unique hom (R-map) $g: F \to M$ with $g(b) = \gamma(b)$ $\forall\, b \in B$.

Too much text: we have diagram:

$$\begin{tikzcd} F \arrow[dr, dashed, "g"] & \\ B \arrow[u, "\text{as basis}"] \arrow[r, "\gamma"', "\text{(in Set)}"'] & M \end{tikzcd}$$

F is the free-est (biggest, tallest) module with a map naming elements for each $b \in B$.

pf. $\forall\ v \in F$, v has a unique expression of the form

$$v = \sum_{b \in B} r_b b \quad \text{(only finitely many } r_b \text{ are nonzero)}$$

Define $\quad g(v) = \sum_{b \in B} r_b \gamma(b)$.

Uniqueness: Suppose g' also fits in the diagram. Then $g'(b) = \gamma(b)\ \forall\ b \in B$; since g' agrees w/ g on a generating set, $g' = g$ on all of F. ∎

For your culture...

Prop. If R is a nonzero $\overset{k}{\text{commutative}}$ ring, then any two bases B, B' of a free module F have the same cardinality.

Q. Have you seen this before?

Pf. idea: m = maximal ideal; then R/m is a field. Then F/mF is a VS over R/m; any two bases of a VS have the same size.

(need comm've R so that we can mod out by the twosided m, and to even get a division ring out!) //

Therefore we can

① define the rank of a free $\overset{\text{comm.}}{k}$-module to be this cardinality

② say $F \cong F'$ iff $\text{rank}_k(F) = \text{rank}_k(F')$

Warning Notion of rank not always defined!

eg. Let k = field, $V \in k\text{-Vect}$.

Then $R = \text{End}_k(V)$

(these are infinite matrices... where columns have finite support...)

$\Rightarrow R \cong R \oplus R$

Recall Presentation of a group?

$\langle$ generators | relations $\rangle$

Here is the module version (remember modules are also abelian groups)

prop. Every $M \in {}_R\mathrm{Mod}$ is a quotient of a free left R-mod F. Moreover, M is finitely generated iff F can be chosen to be finitely generated.

pf. ...

(proof not structured as the prop suggests — b/c its a proof sketch...)

① Given M, define F to be the free module generated by basis $(x_m)_{m \in M}$. everything is a basis element!

$\Rightarrow$ get an R-map $g: F \to M$, $x_m \mapsto m$. clearly surjective!

$\Rightarrow M \cong F/\ker g$.

② If $M = \langle m_1, \dots, m_n \rangle$ is finitely generated (f.g.), then we can choose F to be generated by $(x_i)_{i=1}^n$ where $g: F \longrightarrow M$, $x_i \longmapsto m_i$.

Again, $M \cong F/\ker g$. $\boxtimes$

defn. let $B = (b_i)_{i \in I}$ be a basis for a free left R-mod F.

let $Y = (\sum_i r_{ji} x_i)_{j \in J}$ be a subset of F.

let K = submodule generated by Y.

$\Rightarrow M = F/K$ has the <u>R-module presentation</u>

$$(B \mid Y)$$

generators relations

Basis-free property of free modules (or VS...):

Thm. $R \rightsquigarrow$ a ring, $F \rightsquigarrow$ a free left R-mod.
For any surjection $p: A \twoheadrightarrow A''$
(notation from $0 \to A' \to A \to A'' \to 0$)
and each $h: F \to A''$, there exists a hom $g: F \to A''$
making the following diagram commute:

$$\begin{tikzcd} & F \arrow[dl, dashed, "\exists\, \tilde{h}"'] \arrow[d, "h"] & \\ A \arrow[r, two heads] & A'' \arrow[r] & 0 \end{tikzcd}$$

every $h: F \to A''$ admits a lift $\tilde{h}$
(book: "lifting")

Pf idea

Want to define $\tilde{h}$.

$$\begin{tikzcd} & F \arrow[d, "h"] & (b_i)_{i \in I}. \\ A \arrow[r, two heads, "p"'] & A'' & \end{tikzcd}$$

$b_i \mapsto h(b_i)$

$h(b_i) = p(a_i)$

$a_i \longmapsto p(a_i)$

Let $\tilde{h}(b_i) = a_i$

This defines a module map.
(extend linearly)

Check commutativity of diagram. ✓. //

Rmk The lift $\tilde{h}$ is not necessarily unique because we can choose any $a_i' \in a_i + \ker p$, for instance.

This is a nice basis-free property, indicating we want to study this class of modules

(And they are indeed useful when we talk about constructions that inherently involve a surjection, eg. $\otimes$)

<u>defn.</u> A left-R-mod P is <u>projective</u> if whenever p is surjective, h is any module map, there is a lift $\tilde{h}$:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists \tilde{h}"'] \arrow[d] & \\ A \arrow[r, two heads] & A'' \arrow[r] & 0 \end{tikzcd}$$

<u>Q.</u> Why do we not discuss

$$\begin{tikzcd} & M \arrow[d] \arrow[dr, dashed, "?"] & \\ 0 \arrow[r] & A' \arrow[r] & A \end{tikzcd}$$

<u>Rmk.</u> If "surjection" makes sense in $\mathcal{C}$, then you can define projective objects.

eg. What are the projectives in Groups?

ex. Free only (why?)

$\Longrightarrow$ we can define "free group" w/o respect to a basis.

<u>Q.</u> When do we have proj $\not\Rightarrow$ free?

Turns out this defn. of projective mod isn't good for determining whether M is proj as an R-mod...

Characterizing Projective Modules

Will cover more on Friday, + examples. Here's one way:

<u>Recall</u> The functor $\mathrm{Hom}_R(M, \bullet)$ is <u>left</u>-<u>exact</u>, i.e.

If $0 \to A' \xrightarrow{i} A \xrightarrow{p} A''$ is exact

then $0 \to \mathrm{Hom}_R(M, A') \xrightarrow{i_*} \mathrm{Hom}_R(M, A) \xrightarrow{p_*} \mathrm{Hom}_R(M, A'')$

is exact

<u>Do you recall?</u>

① The <u>functor</u> $\mathrm{Hom}_R(M, *) = \mathrm{Hom}_{R\text{-mod}}(M, *)$

"R-linear" hom

takes R-modules $\longrightarrow$ $Ab = \mathbb{Z}\text{-mod}$

<u>Functorial</u>:

$$\begin{tikzcd} N \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_R(M, N) \arrow[d, "f_*"] \\ N' \arrow[r] & \mathrm{Hom}_R(M, N') \end{tikzcd}$$

② Check all the claims:

- i_* is injective
- $\ker p_* = \operatorname{im} i_*$

A: B/c $\mathrm{Hom}_R(M, -)$ is left exact:

$$\begin{tikzcd} & M \arrow[d, "f"'] \arrow[dr, dashed, "\tilde{f}"] & \\ 0 \arrow[r] & A' \arrow[r, "i"'] & A \end{tikzcd}$$

clearly $\tilde{f}$ exists!

Characterization by the covariant Hom functor:

<u>prop.</u> A left R-mod P is projective iff $\mathrm{Hom}(P, \bullet)$ is an exact (ie left + right exact) functor.

Pf.

Assume $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ exact.

$\mathrm{Hom}_R(P, -)$ is already left exact.

$\Rightarrow$ i_* is injective, $\ker p_* = \operatorname{im} i_*$.

⊛ in practice, just show $\mathrm{Hom}_R(P, -)$ preserves surjectivity!

It remains to show (IRTS) that p_* is surjective:

ie. $\to \mathrm{Hom}_R(P, A) \xrightarrow{p_*} \mathrm{Hom}_R(P, A'') \to 0$ is exact.

Q. What is p_*?

Given $f \in \mathrm{Hom}_R(P, A)$, $\quad P \xrightarrow{f} A$

$p_*(f) = pf$: $\quad P \xrightarrow{f} A \xrightarrow{p} A''$

Now say we're given $h \in \mathrm{Hom}_R(P, A'')$.

We WTF (want to find) $\tilde{h} \in \mathrm{Hom}_R(P, A)$ such that

$p_*(\tilde{h}) = h$ ie. $p\tilde{h} = h$.

Well p is surjective so we're done... here it is:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists \tilde{h}"'] \arrow[d, "h"] & \\ A \arrow[r, "p"'] & A'' \arrow[r] & 0 \end{tikzcd}$$

This characterization shows why projective modules will be useful! If everything were free, homological algebra wouldn't be so interesting.

Next time: Characterization as direct summand of free module.

For now some examples:

eg. Projective but not free module:

Consider $(R\times S)$ modules, when $R, S \neq 0$ are rings.

Then $R \cong R\times\{0\}$ and $S \cong \{0\}\times S$ are

$(R\times S)$-modules.

Projective but not free $R\times S$ modules.

eg. Concrete Consider $M_{2\times 2}(\mathbb{C}) = R$.

Consider $\mathbb{C}^2$ as an R-mod:

indeed, elements $A \in M_{2\times 2}(\mathbb{C})$ act on

2d vectors!

① $\mathbb{C}^2$ is a projective $M_{2\times 2}(\mathbb{C})$-mod:

② But $\mathbb{C}^2$ is not free

Reason

$M_{2\times 2}(\mathbb{C})$ has dim 4 over $\mathbb{C}$, where as

$\mathbb{C}^2$ only is dim

* This may be not very rigorous until we talk about base change...