

Lecture 4 HW: Q: finite G for $kG \cong kG$? as ring?

I'm away Friday!
HW will still be posted

We will return to cat. theory concepts as needed.

§6.4 Free + Projective Modules

defn $F \in R\text{Mod}$ is free if $F \cong \bigoplus_{i \in I} R_i$ where $R_i = R\langle b_i \rangle \cong R$.

direct sum!

named bases element.

i.e. free b/c there are no relations among the b_i .

note. I could be any indexing set.

eg. $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}$, \mathbb{Z}^r ; any vector space V over field k .

Q. ∞ -dual VS?

We could alternatively define free left R -module

in terms of a universal property:

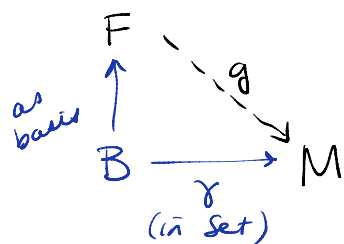
prop. let F be a free left R -mod w/ basis B .

Then $\forall M \in R\text{Mod}$ and set map (function) $\gamma: B \rightarrow M$

there exists a unique

hom (R -map) $g: F \rightarrow M$ with $g(b) = \gamma(b) \quad \forall b \in B$.

Too much text: we have diagram:



F is the free-est (biggest, tallest) module with a map naming elements for each $b \in B$.

pf. $\forall v \in F$, v has a unique expression of the form

$$v = \sum_{b \in B} r_b b \quad (\text{only finitely many } r_b \text{ are nonzero})$$

Define $g(v) = \sum_{b \in B} r_b \gamma(b)$.

Uniqueness: Suppose g' also fits in the diagram. Then

$g'(b) = \gamma(b) \quad \forall b \in B$; since g' agrees w/ g on a generating set, $g' = g$ on all of F . ▣

For your culture...

prop. If R is a nonzero commutative ring, then any two bases B, B' of a free module F have the same cardinality.

Q. Have you seen this before?

Pf. idea: $m = \text{maximal ideal}$; then R/m is a field. Then

F/mF is a VS over R/m ; any two bases of a VS have the same size.

(need comm'ive R so that we can mod out by the two-sided m , and to even get a division ring out!) //

Therefore we can

① define the rank of a free k -module to be this cardinality

② say $F \cong F'$ iff $\text{rank}_k(F) = \text{rank}_k(F')$.

Warning Notion of rank not always defined!

eg. let $k = \text{field}$, $V \in k\text{-vect.}$

Then $R = \text{End}_k(V)$

(these are infinite matrices... where columns have finite support...)

$\Rightarrow R \cong R \oplus R$

Recall Presentation of a group?

$\langle \text{generators} \mid \text{relations} \rangle$

Here is the module version (remember modules are also abelian groups)

prop. Every $M \in {}_R\text{Mod}$ is a quotient of a free left R -mod F .

Moreover, M is finitely generated iff

F can be chosen to be finitely generated.

pf. ...

(proof not structured as the prop suggests - b/c it's a proof sketch...)

① Given M , define F to be the free module generated by basis $(x_m)_{m \in M}$. *everything is a basis element!*

\Rightarrow get an R -map $g: F \rightarrow M$ clearly surjective!
 $x_m \mapsto m$.

$\Rightarrow M \cong F/\ker g$.

② If $M = \langle m_1, \dots, m_n \rangle$ is finitely generated (f.g.),

then we can choose F to be generated by $(x_i)_{i=1}^n$

where $g: F \rightarrow M$
 $x_i \mapsto m_i$.

Again, $M \cong F/\ker g$.



def. let $B = (b_i)_{i \in I}$ be a basis for a free left R -mod F .

let $Y = (\sum_i r_{ji} x_i)_{j \in J}$ be a subset of F .

let $K =$ submodule generated by Y .

$\Rightarrow M = F/K$ has the R -module presentation

$$\begin{array}{ccc} & (B \mid Y) & \\ & \uparrow \quad \uparrow & \\ \text{generators} & & \text{relations} \end{array}$$

Basis-free **property** of free modules (or VS...):

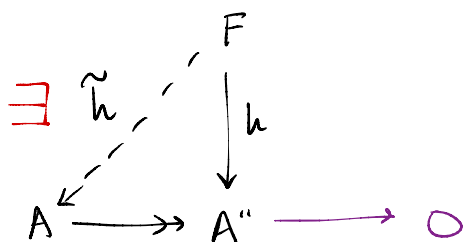
thm. $R \rightsquigarrow$ a ring, $F \rightsquigarrow$ a **free** left R -mod.

For any surjection $p: A \twoheadrightarrow A''$

(notation from $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$)

and each $h: F \rightarrow A''$, there exists a hom $g: F \rightarrow A'$

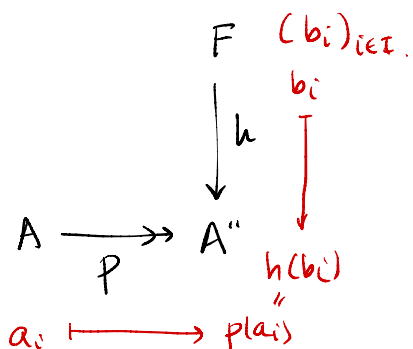
making the following diagram commute:



every $h: F \rightarrow A''$
admits a lift \tilde{h}
(book: "lifting")

Pf idea

Want to define \tilde{h} .



let $\tilde{h}(b_i) = a_i$

This defines a module map.
(extend linearly).

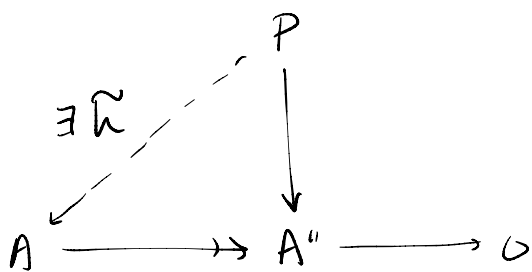
Check commutativity of diagram. \checkmark \equiv

Remark. The lift \tilde{h} is not necessarily unique because we can choose any $a_i' \in a_i + \ker p$, for instance.

This is a nice basis-free property, indicating we want to study this class of modules

(And they are indeed useful when we talk about constructions that inherently involve a surjection, eg. \otimes)

defn. A left- R -mod P is projective if whenever p is surjective, h is any module map, there is a left \tilde{h} :



Q. Why do we not discuss



Remk. If "surjection" makes sense in \mathcal{C} , then you can define projective objects.

eg. What are the projectives in Groups?

ex. Free only (why?)

\Rightarrow we can define "free group" w/o respect to a basis.

Q. When do we have proj \neq free?

Turns out this defn of projective mod isn't good for determining whether M is proj as an R -mod...

Characterizing Projective Modules

Will cover more on Friday, + examples. Here's one way:

Recall The functor $\text{Hom}_R(M, \bullet)$ is left-exact, i.e.

$$\text{If } 0 \rightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \text{ is exact}$$

$$\text{then } 0 \rightarrow \text{Hom}_R(M, A') \xrightarrow{i_*} \text{Hom}_R(M, A) \xrightarrow{p_*} \text{Hom}_R(M, A'')$$

is exact

Do you recall?

① The functor $\text{Hom}_R(M, \bullet) = \text{Hom}_{R\text{-mod}}(M, \bullet)$
 \uparrow
 "R-linear" hom

takes $R\text{-modules} \rightarrow \text{Ab} = \mathbb{Z}\text{-mod}$

Functorial:

$$\begin{array}{ccc} N & \longrightarrow & \text{Hom}_R(M, N) \\ \downarrow f & \circlearrowleft & \downarrow f_* \\ N' & \longrightarrow & \text{Hom}_R(M, N') \end{array}$$

② Check all the claims:

- i_* is injective
- $\ker p_* = \text{im } i_*$

A: B/c $\text{Hom}_R(M, \rightarrow)$ is left exact:

$$\begin{array}{ccc} M & & \\ \downarrow f & \searrow \tilde{f} & \\ 0 \rightarrow A' & \xrightarrow{i} & A \end{array}$$

clearly \tilde{f} exists!

Characterization by the covariant Hom functor:

prop: A left R -mod P is projective iff $\text{Hom}(P, \bullet)$ is an **exact** (i.e. left + right exact) functor.

Pf. Assume $0 \rightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \rightarrow 0$ exact.

$\text{Hom}_R(P, -)$ is already left exact.

$\Rightarrow i_*$ is injective, $\ker p_* = \text{im } i_*$.

\oplus in practice, just show $\text{Hom}_R(P, -)$ preserves surjectivity!

It remains to show (IRTS) that p_* is surjective:

i.e. $\rightarrow \text{Hom}_R(P, A) \xrightarrow{p_*} \text{Hom}_R(P, A'') \rightarrow 0$ is exact.

Q. What is p_* ?

Given $f \in \text{Hom}_R(P, A)$, $P \xrightarrow{f} A$

$p_*(f) = pf: P \xrightarrow{f} A \xrightarrow{p} A''$

Now say we're given $h \in \text{Hom}_R(P, A'')$.

We WTF (want to find) $\tilde{h} \in \text{Hom}_R(P, A)$ such that

$$p_*(\tilde{h}) = h \quad \text{i.e. } p\tilde{h} = h.$$

Well P is surjective so we're done... here it is:

$$\begin{array}{ccc} & P & \\ \exists \tilde{h} \swarrow & \downarrow h & \\ A & \xrightarrow{p} & A'' \rightarrow 0 \end{array}$$

This characterization shows why projective modules will be useful! If everything were free, homological algebra wouldn't be so interesting.

Next time: Characterization as direct summand of free module.

For now some examples:

eg. Projective but not free module:

Consider $(R \times S)$ modules, where $R, S \neq 0$ are rings.

Then $R \cong R \times \{0\}$ and $S \cong \{0\} \times S$ are

$(R \times S)$ -modules.

Projective but not free $R \times S$ modules.

eg. Concrete Consider $M_{2 \times 2}(\mathbb{C}) = R$.

Consider \mathbb{C}^2 as an R -mod:

indeed, elements $A \in M_{2 \times 2}(\mathbb{C})$ act on

2d vectors!

① \mathbb{C}^2 is a projective $M_{2 \times 2}(\mathbb{C})$ -mod:

② But \mathbb{C}^2 is not free

Reason

$M_{2 \times 2}(\mathbb{C})$ has dim 4 over \mathbb{C} , where as

\mathbb{C}^2 only is dim

* This may be not very rigorous until we talk about base change...