

Lecture 5 Colby cover: Olson 158, Friday 1/19, 2:10-3:00 pm.

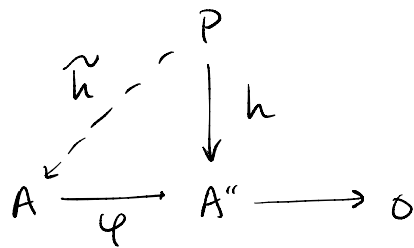
Today: Characterizing projective modules (4 equivalent definitions)
+ more examples

Recall I'll stop writing left R -mod. / right R -mod. We will assume we're working with a particular fixed category of modules. eg. $R\text{Mod}$.

Theorem The following are equivalent: (TFAE)

* Keep this on a board

- ① P is a projective R -mod
- ② If $A \xrightarrow{\varphi} A'' \rightarrow 0$ is exact (ie if φ is surjective) then for all $h \in \text{Hom}_R(P, A'')$, there exists a lift \tilde{h} (making the diagram commute:)



② If $0 \rightarrow A' \xrightarrow{i} A \xrightarrow{\varphi} A'' \rightarrow 0$ is exact, then $0 \rightarrow \text{Hom}(P, A') \xrightarrow{i_*} \text{Hom}(P, A) \xrightarrow{\varphi_*} \text{Hom}(P, A'') \rightarrow 0$ is also exact.

③ If $P \cong M/K$ then P is \cong to a submodule of M , ie. every short exact sequence

$$0 \rightarrow K \rightarrow M \rightarrow P \rightarrow 0 \text{ splits .}$$

Do you remember what this means?

④ P is a direct summand of a **free** R -module.

Recall Covered in 250B; quick review

① A SES splits if the dotted map exists: (a 'section')

$$0 \longrightarrow K \xrightarrow{i} M \xrightarrow{p} Q \longrightarrow 0 \quad \text{where } ps = id_Q.$$

"kernel" *"quotient"*

② If the SES splits, then $M \cong K \oplus Q$

Rough idea

$$0 \longrightarrow K \xrightarrow{i} M \xrightarrow{p} Q \longrightarrow 0$$

k $m = i(k) + s(p(m))$ $p(m)$

Pf. Define an isomorphism

$$\psi: K \oplus Q \longrightarrow M$$

$$(k, q) \longmapsto i(k) + s(q)$$

$$\varphi: M \longrightarrow K \oplus Q$$

Let $m \in M$.

Then $p(m) \in Q$ corresponds to the coset $m + K$.

Consider $sp(m)$; since $ps = id_Q$, $psp(m) = p(m)$,

so $sp(m) \in m + K$. Let $m_Q = sp(m)$.

Then $m_K := m - sp(m) \in K$.

Then $m = m_K + m_Q$.

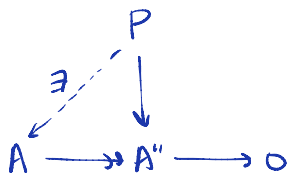
Check that $\varphi\psi = id_M$, $\psi\varphi = id_{K \oplus Q}$.

no need
to over
if
students
recall
split
SES.

Pf. We talked about ①, ①, ② last time.
 (① is the definition of ① in Rotman.)

$$\boxed{② \Rightarrow ③}$$

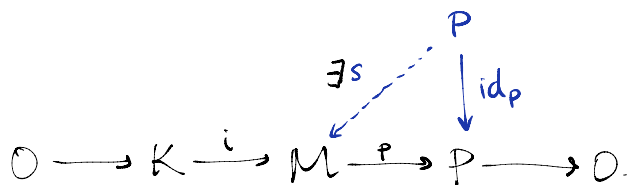
Assume:



Suppose we have an exact sequence

$$0 \rightarrow K \xrightarrow{i} M \xrightarrow{p} P \rightarrow 0.$$

Consider $A'' = P$. Then



and $ps = id_P \Rightarrow s$ is a section.

③ \Rightarrow ④

Lemma. Every module is the quotient of a free module:
(In lecture 4 notes; we didn't cover this in class)

pf.

Given M , define F to be the free module generated by
basis $(x_m)_{m \in M}$. *everything is a basis element!*

\Rightarrow get an R -map $g: F \rightarrow M$ clearly surjective!
 $x_m \mapsto m$.

$\Rightarrow M \cong F/\ker g$. //

Consider any free F such that P is a quotient of F .

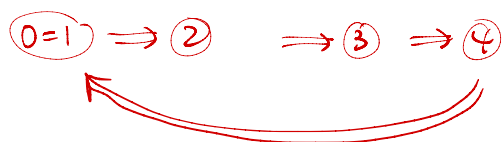
Let $\pi: F \twoheadrightarrow P$ be the quotient map, and let $K = \ker \pi$.

\Rightarrow get the SES

$$0 \rightarrow K \xrightarrow[\text{inclusion}]{i} F \xrightarrow{\pi} P \rightarrow 0.$$

By assumption (③), this splits, so $F \cong K \oplus P$. //

To complete the TFAE proof, we need to complete the cycle
of implications

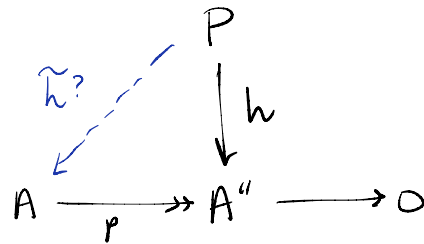


④ \Rightarrow ①: *The interesting one!* Go more slowly here

Suppose $F \cong P \oplus K$, where F is the free module generated by basis set B .

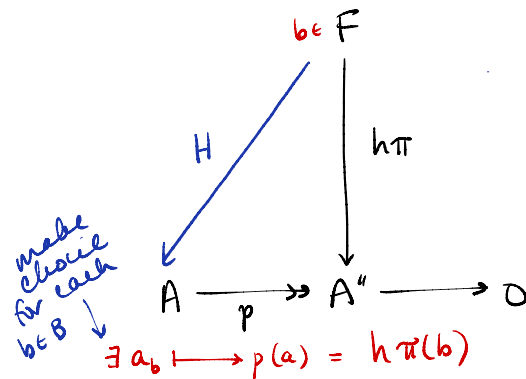
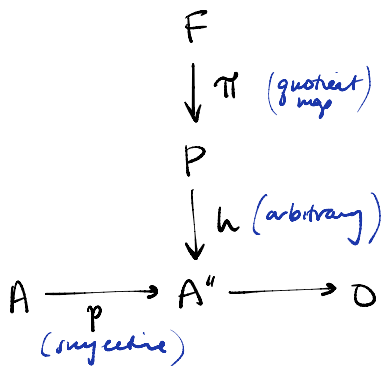
Suppose we are given

(We a priori don't know if P is projective yet — but we do know free modules are projective!)



We WTS a lift \tilde{h} exists.

Consider



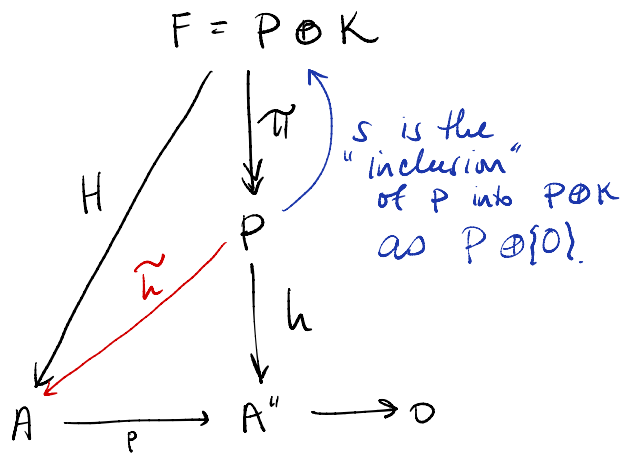
Define H by $H(b) = a_b$, extend linearly.

(After the choices $\{a_b\}$ are made, the map H exists and is unique by the univ prop of free modules)

But $F \cong P \oplus K$! So define $\tilde{h}(x) = H((x, 0))$.

(for easier notation we treat this as an equal sign)

Then



"clearly" commutes (because $\tilde{h} = sH$)

\Rightarrow a lift \tilde{h} exists, so P is indeed projective. \square

Time for examples, now that we have 4 ways of determining whether an R -mod is projective

nontrivial rings. i.e. $1 \neq 0$

eg. let $R, S \neq 0$ be rings, and consider them as left $(R \times S)$ -modules.

ie (r, s) acts on R by left mult by r ,

and the s does nothing (ie s acts by 1_R).

Then both R and S are projective

(they are summands of the free module $R \times S$)

but they are not free:

eg. More concrete example: Consider $M_{2 \times 2}(\mathbb{C}) = R$.

Consider \mathbb{C}^2 as an R -mod.

indeed, elements $A \in M_{2 \times 2}(\mathbb{C})$ act on \mathbb{C}^2 vectors!

① \mathbb{C}^2 is a projective $M_{2 \times 2}(\mathbb{C})$ -mod:

b/c $\mathbb{C}^2 \cong$ submodule of diagonal matrices in the free module R

② But \mathbb{C}^2 is not free

Reason $M_{2 \times 2}(\mathbb{C})$ has dim 4 over \mathbb{C} , whereas

\mathbb{C}^2 only is dim

* This may be not very rigorous until we talk about base change...