Lecture 5 Colby cover: Olson 158, Friday 1/19, 2:10—3:00 pm.

Today: Characterizing projective modules (4 equivalent definitions)

+ more examples

Recall I'll stop writing left R-mod. / right R-mod. We will assume we're working with a particular fixed category of modules. eg. ${}_R\mathrm{Mod}$.

Theorem The following are equivalent: (TFAE)

* Keep this on a board

⓪ P is a projective R-mod

① If $A \xrightarrow{\varphi} A'' \longrightarrow 0$ is exact (ie if φ is surjective)

then for all $h \in \mathrm{Hom}_R(P, A'')$, there exists a lift

$\tilde{h}$ (making the diagram commute :)

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{h}"'] \arrow[d, "h"] & \\ A \arrow[r, "\varphi"'] & A'' \arrow[r] & 0 \end{tikzcd}$$

② If $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0$ is exact, then

$$0 \longrightarrow \mathrm{Hom}(P, A') \xrightarrow{i_*} \mathrm{Hom}(P, A) \xrightarrow{p_*} \mathrm{Hom}(P, A'') \longrightarrow 0$$

is also exact.

③ If $P \cong M/K$ then P is $\cong$ to a submodule of M.

ie. every short exact sequence

$0 \to K \to M \to P \to 0$ splits. Do you remember what this means?

④ P is a direct summand of a free R-module.

Recall   Covered in 250B; quick review

① A SES splits if the dotted map exists: (a "section")

$$0 \longrightarrow K \xrightarrow{i} M \underset{\curvearrowleft s}{\xrightarrow{p}} Q \longrightarrow 0 \qquad \text{where } ps = id_Q.$$

"kernel"   "quotient"

② If the SES splits, then $M \cong K \oplus Q$

Rough idea

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & M \arrow[r, "p"] & Q \arrow[r] \arrow[l, bend right, "s"'] & 0 \end{tikzcd}$$

$$k \qquad m = i(k) + s(p(m)) \qquad p(m)$$

Pf. Define an isomorphism

$$\varphi : K \oplus Q \longrightarrow M$$
$$(k, q) \longmapsto i(k) + s(q)$$

$$\psi : M \longrightarrow K \oplus Q$$

Let $m \in M$.

Then $p(m) \in Q$ corresponds to the coset $m + K$.

Consider $sp(m)$; since $ps = id_Q$, $psp(m) = p(m)$,

so $sp(m) \in m + K$. Let $m_Q = sp(m)$.

Then $m_K := m - sp(m) \in K$.

Then $m = m_K + m_Q$.

no need to cover if students recall split SES.

Check that $\psi\varphi = id_M$, $\varphi\psi = id_{K\oplus Q}$.

Pf. We talked about ⓪, ①, ② last time.
(① is the definition of ⓪ in Rotman.)

$\boxed{② \Rightarrow ③}$

Assume:

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dashed, "\exists"'] & \\ A \arrow[r, two heads] & A'' \arrow[r] & 0 \end{tikzcd}$$

Suppose we have an exact sequence

$$0 \longrightarrow K \xrightarrow{i} M \xrightarrow{p} P \longrightarrow 0.$$

Consider $A'' = P$. Then

$$\begin{tikzcd} & & & P \arrow[d, "id_P"] \arrow[dl, dashed, "\exists s"'] & \\ 0 \arrow[r] & K \arrow[r, "i"] & M \arrow[r, "p"] & P \arrow[r] & 0. \end{tikzcd}$$

and $ps = id_P \Rightarrow s$ is a section.

③ ⇒ ④

<u>Lemma</u>. Every module is the quotient of a free module.
(In lecture 4 notes; we didn't cover this in class)

Pf.
Given M, define F to be the free module generated by basis $(x_m)_{m \in M}$. everything is a basis element!

$\Rightarrow$ get an R-map $\quad g: F \to M$, $\; x_m \mapsto m$. $\quad$ clearly surjective!

$\Rightarrow M \cong F/\ker g$. //

Consider any free F such that P is a quotient of F.
Let $\pi: F \twoheadrightarrow P$ be the quotient map, and let $K = \ker \varphi$.
$\Rightarrow$ get the SES

$$0 \longrightarrow K \xrightarrow[\text{inclusion}]{i} F \xrightarrow{\pi} P \longrightarrow 0.$$

By assumption (③), this splits, so $F \cong K \oplus P$. //

To complete the TFAE proof, we need to complete the cycle of implications

$$\begin{tikzcd} (0=1) \arrow[r, Rightarrow] & (2) \arrow[r, Rightarrow] & (3) \arrow[r, Rightarrow] & (4) \arrow[lll, Rightarrow, bend left] \end{tikzcd}$$

④ ⇒ ①: The interesting one! Go more slowly here

Suppose $F \cong P \oplus K$, where F is the free module generated by basis set B.

Suppose we are given

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{h}?"'] \arrow[d, "h"] & \\ A \arrow[r, two heads, "p"'] & A'' \arrow[r] & 0 \end{tikzcd}$$

(We a priori don't know if P is projective yet — but we do know free modules are projective!)

We WTS a lift $\tilde{h}$ exists.

Consider

$$\begin{tikzcd} & F \arrow[d, "\pi\ (\text{quotient map})"] & \\ & P \arrow[d, "h\ (\text{arbitrary})"] & \\ A \arrow[r, "p"', "(\text{surjective})"'] & A'' \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} & b \in F \arrow[dl, "H"'] \arrow[d, "h\pi"] & \\ A \arrow[r, two heads, "p"'] & A'' \arrow[r] & 0 \end{tikzcd}$$

make choice for each $b \in B$

$\exists\, a_b \longmapsto p(a) = h\pi(b)$

Define H by $H(b) = a_b$, extend linearly.

(After the choices $\{a_b\}$ are made, the map H exists and is unique by the univ prop of free modules)

But $F \cong P \oplus K$! So define $\tilde{h}(x) = H((x, 0))$.

(for easier notation we treat this as an equal sign)

Then

$$
\begin{tikzcd}
 & F = P \oplus K \arrow[dl, "H"'] \arrow[d, "\pi"] \\
 & P \arrow[u, bend right] \arrow[dl, "\tilde{h}"'] \arrow[d, "h"] \\
A \arrow[r, "p"'] & A'' \arrow[r] & 0
\end{tikzcd}
$$

s is the "inclusion" of P into $P \oplus K$ as $P \oplus \{0\}$.

"clearly" commutes (because $\tilde{h} = sH$)

$\Rightarrow$ a lift $\tilde{h}$ exists, so P is indeed projective. ∎

Time for examples, now that we have 4 ways of determining whether an R-mod is projective

eg. Let $\underbrace{R, S \neq 0}$ be rings, and consider them as left $(R \times S)$-modules.

nontrivial rings. ie $1 \neq 0$

ie (r,s) acts on R by left mult by r, and the s does nothing (ie s acts by 1_R).

Then both R and S are projective

(they are summands of the free module $R \times S$)

but they are not free:

eg. More concrete example: Consider $M_{2\times2}(\mathbb{C}) = R$.

Consider $\mathbb{C}^2$ as an R-mod.

indeed, elements $A \in M_{2\times2}(\mathbb{C})$ act on 2d vectors!

① $\mathbb{C}^2$ is a projective $M_{2\times2}(\mathbb{C})$-mod:

b/c $\mathbb{C}^2 \cong$ submodule of diagonal matrices in the free module R

② But $\mathbb{C}^2$ is not free

Reason $M_{2\times2}(\mathbb{C})$ has dim 4 over $\mathbb{C}$, whereas $\mathbb{C}^2$ only is dim

* This may be not very rigorous until we talk about base change...