Lecture 6 Injective Modules

I'm done w/ Rotman's notation; will use mine instead.

- last week, we saw how projective modules are exactly those $P \in {}_R\mathrm{Mod}$ that make the functor $\mathrm{Hom}_R(P, -)$ exact.
- $\mathrm{Hom}_R(M, -)$ is a <u>covariant</u> functor:

If $A \xrightarrow{\varphi} B$

then the induced map

$$\begin{array}{ccc} \mathrm{Hom}_R(M, A) & \longrightarrow & \mathrm{Hom}_R(M, B) \\ [f : M \to A] & \longmapsto & [\varphi f : M \to B] \end{array}$$

points in the same direction.

<u>defn.</u> let $\mathcal{C}, \mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}$ functor.

Let $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ where $A, B \in \mathcal{C}$.

① If $F(\varphi) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$, then F is <u>covariant</u>.

② If $F(\varphi) \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$, then F is <u>contravariant</u>.

<u>eg.</u> $\mathrm{Hom}_R(-, M)$ * note these are R-maps <u>to</u> M.

Let $A \xrightarrow{\varphi} B$ be an R-module map. $\varphi \in \mathrm{Hom}_R(A, B)$

Now consider

upper star to indicate induced map when using contravariant Hom

$$\begin{array}{ccc} \mathrm{Hom}_R(A, M) & \xleftarrow{\varphi^*(g)} & \mathrm{Hom}_R(B, M) \\ [g\varphi : A \to M] & \longleftarrow & [g : B \to M] \end{array}$$

$\varphi^*(g) = g\varphi$, by precomposition: $A \xrightarrow{\varphi} B \xrightarrow{g} M$

We will use this Hom functor to define <u>injective</u> modules.

Exercise $\mathrm{Hom}_R(-,M)$ is **left exact**: same as $\mathrm{Hom}_R(M,-)$

If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact,

then $0 \to \mathrm{Hom}_R(C,M) \xrightarrow{\beta^*} \mathrm{Hom}_R(B,M) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A,M)$

is exact. (HW03)

defn. $Q \in {}_R\mathrm{Mod}$ is *injective* if

for any $A, B \in {}_R\mathrm{Mod}$ and injective map $\alpha : A \hookrightarrow B$

ie $0 \to A \xrightarrow{\alpha} B$ is exact,

and for any $f : A \to Q$,

there exists a lift $\tilde{f}$ making the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists \tilde{f}"] \\
& Q &
\end{tikzcd}$$

While injective modules are not as easy to characterize as projective modules, we still have three equivalent definitions.

prop Let $Q \in {}_R\mathrm{Mod}$. TFAE

① Q is injective by the defn above

② $\mathrm{Hom}_R(-,Q)$ is exact

HW03: can only use these as defn!

③ If Q is a submodule of $M \in {}_R\mathrm{Mod}$, then Q is a direct summand of M. In other words, any exact sequence $0 \to Q \to M \to N \to 0$ splits.

Maybe discuss after Baer's criterion

Cor. $\mathrm{Hom}_R(-,Q)$ is exact iff Q is injective.

Pf.

① $\Rightarrow$ ② $\Leftrightarrow$ (actually) (Actually a short proof, but let's recall what's already known.)

Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be exact.

Recall that $\mathrm{Hom}_R(-,Q)$ is left exact. (HW03)

So ISTS that $\mathrm{Hom}_R(-,Q)$ is right exact.

We only need to show exactness here

WTS the following sequence is exact. show exactness here on HW03!

$$\mathrm{Hom}_R(C,Q) \xrightarrow{\beta^*} \mathrm{Hom}_R(B,Q) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A,Q) \longrightarrow 0$$

$$[g: C \to Q] \longmapsto [g\beta : B \to C \to Q]$$

$$[h: B \to Q] \longmapsto [h\alpha : A \to B \to Q]$$

Let $f \in \mathrm{Hom}_R(A,Q)$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists \tilde{f}"] \\ & Q & \end{tikzcd}$$

By assumption, $\exists\, \tilde{f} \in \mathrm{Hom}_R(B,Q)$ s.t. $\tilde{f}\alpha = f$.

$\Rightarrow \alpha^*$ is surjective.

maybe after Baer.

② $\Rightarrow$ ③ Assume $\mathrm{Hom}_R(-,Q)$ is exact; WTS any $0 \to Q \to M \to N \to 0$ splits.

Let $0 \to Q \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ be an exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r, "\iota"] \arrow[d, "f = \mathrm{id}_Q"'] & M \arrow[dl, dashed, "\exists \tilde{f}"] \\ & Q & \end{tikzcd}$$

$\Rightarrow \tilde{f}\iota = \mathrm{id}_Q$.

$\Rightarrow \tilde{f}$ is a splitting homomorphism $\Rightarrow$ SES splits.

(Recall split SES?) (left split / right split - splitting lemma?)

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r, "\iota"] & M \arrow[r, "\pi"] \arrow[l, bend right, "\tilde{f}"'] & N \arrow[r] & 0 \end{tikzcd}$$

③ $\Rightarrow$ ① need more tech.

<u>thm.</u> Every module is contained in an injective module.

Compare: every $M \in {}_R\mathrm{Mod}$ is a quotient of a projective (actually free!) module. Prove on HW04! (Start early!)

* there's a notion of the smallest, "best" injective to stick a module in: "injective hull".

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③ ⇒ ① Suppose D is a module s.t. every SES

$$0 \to D \to M \to N \to 0 \text{ splits.}$$

<u>prop.</u> (HW03) $Q_1 \oplus Q_2$ inj iff both Q_i inj.

By thm, $D \subset Q$ where Q is an injective R-mod

$$0 \to D \to Q \to Q/D \to 0 \quad \text{is exact}$$

By assumption, this splits $\Rightarrow$ $Q \cong D \oplus Q/D$

$\Rightarrow D$ is injective.

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note Injective modules are harder to characterize than projective.

When R is a PID, we have some easier criteria to check:

Prop. Let $Q \in {}_R\mathrm{Mod}$.

① (Baer's Criterion) Q is injective iff

$\forall$ left ideal $I \subset R$,

any R-hom $g: I \longrightarrow Q$ can be extended to an R-hom $G: R \longrightarrow Q$.

Recall left ideals $I \in {}_R\mathrm{Mod}$.

② If R is a PID, then Q is injective iff

$rQ = Q$ for every nonzero $r \in R$.

↳ In particular, a $\mathbb{Z}$-module $A \in \mathrm{Ab}$ is injective iff it is divisible, ie $A = nA$ $\forall n \in \mathbb{Z}$ where $n \neq 0$

ie divisible by all reasonable n

③ When R is a PID, quotient modules of injective R-mods are also injective.

Before we discuss the proof, consider examples:

① $\mathbb{Z}$ is not divisible $\Rightarrow$ not injective as $\mathbb{Z}$-mod.

($\Rightarrow$ free modules are not necessarily injective!)

② $\mathbb{Q}$ is divisible, as are all quotients of $\mathbb{Q}$, eg. $\mathbb{Q}/\mathbb{Z}$.

③ $\bigoplus$ of divisible $\mathbb{Z}$-mods is divisible ($\Rightarrow$ $\bigoplus$ of inj $\mathbb{Z}$-mods is inj)

④ Turns out no nonzero finitely generated $\mathbb{Z}$-mod is injective.

(use classification) HW04

⑤ If $R = \mathbb{F}$ a field, then every $\mathbb{F}$-mod (ie $\mathbb{F}$-VS) is injective

Cor. Every $\mathbb{Z}$-mod is a submodule of an injective $\mathbb{Z}$-mod.

(Will need this to show every R-mod is a submodule of an injective R-mod).

Pf. Let $M \in \mathbb{Z}$-mod, $\mathcal{B}$ = set of $\mathbb{Z}$-mod generators for M.

Let $\mathcal{F} = F(\mathcal{B})$ = free $\mathbb{Z}$-mod on $\mathcal{B}$.

$\Rightarrow$ SES $\quad 0 \to K \longrightarrow \mathcal{F} \longrightarrow M \longrightarrow 0 \quad$ (identify $M = \mathcal{F}/K$)

Let $\mathcal{Q}$ ← \mathcal{Q} be the free $\mathbb{Q}$-mod on $\mathcal{B}$.

$\Rightarrow \mathcal{Q} \cong \bigoplus_{i \in I} \mathbb{Q} \Rightarrow \mathcal{Q}$ is divisible $\Rightarrow \mathcal{Q}$ is injective.

Note that $\mathcal{Q}$ contains $\mathcal{F}$, which in turn contains K.

$\Rightarrow$ K is also a $\mathbb{Z}$-submodule of $\mathcal{Q} \Rightarrow \mathcal{Q}/K$ is injective (by ③ in prop)

$\Rightarrow M = \mathcal{F}/K \subseteq \mathcal{Q}/K$ where $\mathcal{Q}/K$ is injective. ☺

HW 04: Prove this for general R-modules.

Compare this to: Every R-mod is a quotient of a projective (in fact free) R-mod.

Pf. of prop. (Baer's Criterion, + more) (Next time!)