

Lecture 7

* Make note about $Q_1 \oplus Q_2$ HW problem + $\varphi_1 \oplus \varphi_2$ notation.

* #7 on HW: $\mathbb{I} = \mathbb{N}$ (countable basis!)

* #4: yes can use SES char of inj modules (sorry)

* #8: just give answer - 2 sentences!

Recall

A \mathbb{Z} -mod A is divisible iff $A = nA \quad \forall n \neq 0$.

Last time we stated and discussed:

prop. let $Q \in {}_R\text{Mod}$.

① (Baer's Criterion) Q is injective iff

\forall left ideal $I \subset R$,

any R -hom $g: I \rightarrow Q$ can be extended to

an R -hom $\tilde{g}: R \rightarrow Q$.

} Recall left ideals
 $I \in {}_R\text{Mod}$.

② If R is a PID, then Q is injective iff

$rQ = Q$ for every nonzero $r \in R$.

↳ In particular, a \mathbb{Z} -module $A \in \text{Ab}$ is injective iff

it is divisible, i.e. $A = nA \quad \forall n \in \mathbb{Z}$ where $n \neq 0$

↑ i.e. divisible
by all reasonable
 n

③ When R is a PID, quotient modules of injective R -mods are also injective.

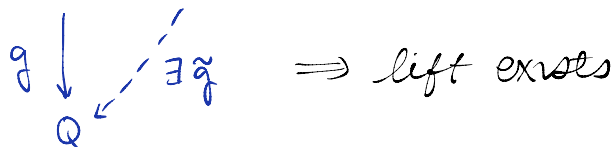
Now that you're excited and alert, here's the proof.

① \Rightarrow (just special case of defn of injective)

Assume: Q is injective

$g: I \rightarrow Q$ is an R -module map (where $I \neq 0$)

Consider the SES $0 \rightarrow I \hookrightarrow R$



\Leftarrow more interesting!

Assume: Every $g: I \rightarrow Q$ lifts to some $\tilde{g}: R \rightarrow Q$.

Consider an exact seqn. $0 \rightarrow L \xrightarrow{i} M$ (view $L \hookrightarrow M$ as inclusion)
and a map $f: L \rightarrow Q$: $\begin{array}{ccc} & & M \\ & \nearrow \text{WTS } \exists \tilde{f} & \\ L & \xrightarrow{f} & Q \end{array}$

Define a partially ordered set S as follows:

$$S = \left\{ (L', f') : \begin{array}{l} L \subset L' \subset M \\ \text{submodule} \end{array}, \begin{array}{ccc} 0 & \rightarrow & L & \rightarrow & L' \\ & & f \downarrow & \nearrow f' & \\ & & Q & & \end{array} \right\}$$

i.e. submodules L' of M that contain L ,

together with an R -map $f': L' \rightarrow Q$ that lifts f .

partial order: $(L_1, f_1) \leq (L_2, f_2)$ iff

$$L_1 \subset L_2, \text{ and } f_2|_{L_1} = f_1$$

i.e. they are related by

$$\begin{array}{ccc} 0 & \rightarrow & L_1 & \xrightarrow{i} & L_2 \\ & & f_1 \downarrow & \nearrow f_2 & \\ & & Q & & \end{array}$$

Recall Zorn's lemma:

Let (X, \leq) be a nonempty poset. If every chain has an upper bound, then X contains (at least one) maximal element.

- Since $(L, f) \in S$, S is nonempty.
- **Why does every chain have an upper bound?**

(Usual argument)

Ans. Let $(L_0, f_0) \leq (L_1, f_1) \leq \dots$ be a chain.

Define (L_∞, f_∞) where $L_\infty = \bigcup_{i=0}^{\infty} L_i$ as a set;
for any $l \in L_\infty$, $l \in L_i$ for some i .

Let $f_\infty(l) = f_i(l)$.

↳ Note that this is well-defined, since

if $l \in L_i$ and L_j , then whether $i < j$ or $j < i$,

$$f_i(l) = f_j(l).$$

(Check that this is indeed a module & module map by thinking through it)

if
families
can be
quicker

⇒ We can apply Zorn's lemma to pick out a maximal element $(L^{\max}, f^{\max}) \in S$.

Now **ISTS** that $L^{\max} = M$. (and then $f^{\max} = f$ as desired).

Claim $L^{\max} = M$.

By way of contradiction (BWOC), suppose $\exists m \in M$ s.t. $m \notin L^{\max}$.

Define $I = \{r \in R \mid rm \in L^{\max}\}$.

Check that this is a left ideal: $RI = I$ (right?)

Define $g: I \rightarrow Q$
 $x \mapsto f^{\max}(xm)$ (Check this is R -map...
 r -action clear...)

By the hypothesis, a lift $\tilde{g}: R \rightarrow Q$ exists.

Consider the submodule $M' = L^{\max} + Rm \subset M$.

& define the map $F': M' \rightarrow Q$
 $(l^{\max} + rm) \mapsto f^{\max}(l^{\max}) + \tilde{g}(r)$.

$L^{\max} + Rm \rightarrow Q$
 $l^{\max} + rm \mapsto f^{\max}(l^{\max}) + \tilde{g}(r)$ } maybe clearer

Claim F' is well-defined

pf.

If $l_1^{\max} + r_1 m = l_2^{\max} + r_2 m$,

then $\underline{(r_1 - r_2)m} \stackrel{\oplus}{=} l_2^{\max} - l_1^{\max} \in \underline{L^{\max}}$

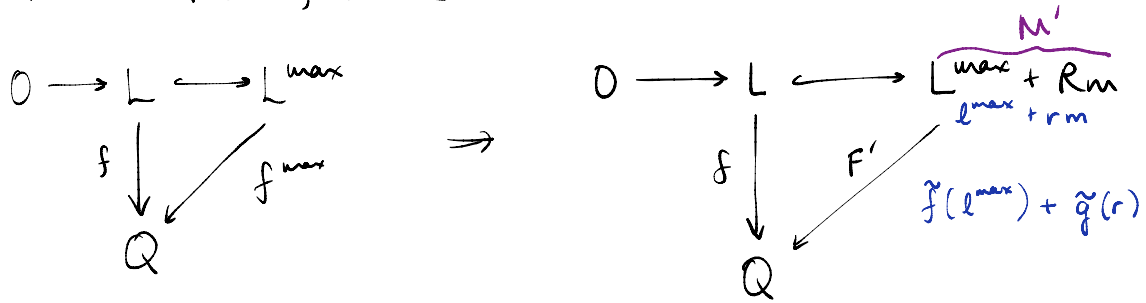
$\Rightarrow r_1 - r_2 \in I$ (by the defn of I)

$\Rightarrow \tilde{g}(\underline{r_1 - r_2}) = \tilde{g}(\underline{r_1 - r_2}) = f^{\max}(\underline{(r_1 - r_2)m}) = f^{\max}(l_2^{\max} - l_1^{\max})$
↑ by defn of \tilde{g} ↑ by \oplus

Therefore $f^{\max}(l_1^{\max}) + \tilde{g}(r_1) = f^{\max}(l_2^{\max}) + \tilde{g}(r_2)$
 $F'(l_1^{\max} + r_1 m) = F'(l_2^{\max} + r_2 m)$

(Check this is indeed an R -map, by inspection)

Note that F' extends f to $L^{\max} + Rm$:



This contradicts maximality of L^{\max} ! $(m \notin L^{\max} \text{ but } m \in L^{\max} + Rm)$

Therefore $L^{\max} = M$, and we can choose $\tilde{f} = f^{\max}$.

Take a breather before we move on to ②, ③, + more, which are really just corollaries and much less intense.

Pf of ②

Assume R is a PID.

Every nonzero ideal $I \subset R$ is of the form $I = (r)$ ($r \neq 0$).
(and conversely, $\forall r \neq 0$, (r) is an ideal.)

Then any R -map $f: I \rightarrow Q$ is determined by
choice of $f(r) = q$.

This homomorphism can be extended to $\tilde{f}: R \rightarrow Q$

iff $\exists q' \in Q$ such that $\tilde{f}(1) = q'$ and

$$q = f(r) = \tilde{f}(r) = r \cdot q'$$

(by R -linearity of \tilde{f})

i.e. we need q' such that $q = r q'$.

(and q was arbitrary,
and there is a hom \tilde{f} for
every choice $q \in Q$, $q = \tilde{f}(r)$!)

Basis criterion is therefore satisfied iff $rQ = Q$.

Pf. of ③

Suppose Q is an inj module over a PID,

i.e. $Q = rQ \forall r \neq 0$ in R .

In a quotient module $\bar{Q} = Q/K$ where $K \subset Q$,

$\bar{Q} = r\bar{Q}$ still:

If $q_1 = r q_2$ then $\bar{q}_1 = r \bar{q}_2 \dots$ $\pi: Q \rightarrow Q/K$ is an R -isompl.

Important Corollary to the Prop:

Cor. Every \mathbb{Z} -mod is a submodule of an injective \mathbb{Z} -mod.

(Will need this to show every R -mod is a submodule of an injective R -mod).

Pf.

Let $M \in \mathbb{Z}$ -mod, $B =$ set of \mathbb{Z} -mod generators for M .

Let $\mathcal{F} = F(B) =$ free \mathbb{Z} -mod on B .

\Rightarrow SES $0 \rightarrow K \rightarrow \mathcal{F} \rightarrow M \rightarrow 0$ (identify $M = \mathcal{F}/K$)

Let $\mathcal{Q} \leftarrow \{\text{mathcal{Q}}\} \mathbb{Q}$ be the free \mathbb{Q} -mod on B .

$\Rightarrow \mathcal{Q} \cong \bigoplus_{i \in I} \mathbb{Q} \Rightarrow \mathcal{Q}$ is divisible $\Rightarrow \mathcal{Q}$ is injective.

Note that \mathcal{Q} contains \mathcal{F} , which in turn contains K .

$\Rightarrow K$ is also a \mathbb{Z} -submodule of $\mathcal{Q} \Rightarrow \mathcal{Q}/K$ is injective (by ③ in prop)

$\Rightarrow M = \mathcal{F}/K \subseteq \mathcal{Q}/K$ where \mathcal{Q}/K is injective. \square

HW04: Prove this for general R -modules.

Compare this to: Every R -mod is a quotient of a projective

(in fact free) R -mod.