Lecture 7

* Make note about $Q_1 \oplus Q_2$ HW problem + $\varphi_1 \oplus \varphi_2$ notation.

* #7 on HW: $I = \mathbb{N}$ (countable basis!)

* #4: yes can uses SES char of inj modules (sorry)

* #8: just quick answer - 2 sentences!

Recall

A $\mathbb{Z}$-mod A is divisible iff $A = nA$ $\forall\, n \neq 0$.

Last time we stated and discussed:

Prop. Let $Q \in {}_R\mathrm{Mod}$.

① (Baer's Criterion) Q is injective iff

$\forall$ left ideal $I \subset R$,

any R-hom $g: I \to Q$ can be extended to an R-hom $\tilde{g}: R \to Q$.

change of notation from previous statement of theorem (now my notation)

Recall left ideals $I \in {}_R\mathrm{Mod}$.

② If R is a PID, then Q is injective iff

$rQ = Q$ for every nonzero $r \in R$.

↳ In particular, a $\mathbb{Z}$-module $A \in \mathrm{Ab}$ is injective iff it is divisible, ie $A = nA$ $\forall\, n \in \mathbb{Z}$ where $n \neq 0$

ie divisible by all reasonable n

③ When R is a PID, quotient modules of injective R-mods are also injective.

Now that you're excited and alert, here's the proof.

① $\boxed{\Rightarrow}$ (just special case of defn of injective)

<u>assume</u>: Q is injective

$g: I \to Q$ is an R-module map (where $I \neq 0$)

Consider the SES

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, "g"'] & R \arrow[dl, dashed, "\exists \tilde{g}"] \\ & Q & \end{tikzcd}$$

$\Rightarrow$ lift exists

$\boxed{\Leftarrow}$ more interesting!

<u>Assume</u>: Every $g: I \to Q$ lifts to some $\tilde{g}: R \to Q$.

Consider an exact seqn. and a map $f: L \to Q$:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "f"'] & M \arrow[dl, dashed, "\text{WTS } \exists \tilde{f}"] \\ & Q & \end{tikzcd}$$

(view $L \hookrightarrow M$ as inclusion)

Define a $\overbrace{\text{partially ordered}}^{\text{"poset"}}$ set S as follows:

$$S = \left\{ (L', f') : \underbrace{L \subset L' \subset M}_{\text{submodule}} , \quad \begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "f"'] & L' \arrow[dl, "f'"] \\ & Q & \end{tikzcd} \right\}$$

ie submodules L' of M that contain L,

together with an R-map $f': L' \to Q$ that lifts f.

partial order: $(L_1, f_1) \leq (L_2, f_2)$ iff

$L_1 \subset L_2$, and $f_2|_L = f_1$

ie. they are related by

$$\begin{tikzcd} 0 \arrow[r] & L_1 \arrow[r, hook, "i"] \arrow[d, "f_1"'] & L_2 \arrow[dl, "f_2"] \\ & Q & \end{tikzcd}$$

Recall Zorn's lemma:

Let $(X, \preceq)$ be a nonempty poset. If every chain has an upper bound, then X contains (at least one) maximal element.

- Since $(L, f) \in S$, S is nonempty.
- Why does every chain have an upper bound?

(Usual argument)

if families can be quick

Ans. Let $(L_0, f_0) \preceq (L_1, f_1) \preceq \ldots$ be a chain.
Define (L_∞, f_∞) where $L_\infty = \bigcup_{i=0}^{\infty} L_i$ as a set;
for any $l \in L_\infty$, $l \in L_i$ for some i.
Let $f_\infty(l) = f_i(l)$.
↳ Note that this is well-defined, since if $l \in L_i$ and L_j, then whether $i<j$ or $j<i$, $f_i(l) = f_j(l)$.

(Check that this is indeed a module & module map by thinking through it)

$\Rightarrow$ We can apply Zorn's lemma to pick out a maximal element $(L^{max}, f^{max}) \in S$.

Now ISTS that $L^{max} = M$. (and then $f^{max} = \tilde{f}$ as desired).

<u>Claim</u> $L^{max} = M$.

By way of contradiction (BWOC), suppose $\exists\ m \in M$ s.t. $m \notin L^{max}$.

Define $I = \{ r \in R \mid rm \in L^{max} \}$.

Check that this is a left ideal: $RI = I$ (right?)

Define $g: I \longrightarrow Q$
$x \longmapsto f^{max}(xm)$ (Check this is R-map... r-action clear ...)

By the hypothesis, a lift $\tilde{g}: R \longrightarrow Q$ exists.

Consider the submodule $M' = L^{max} + Rm \subset M$.

& define the map $F': M' \longrightarrow Q$
$(\ell^{max} + rm) \longmapsto f^{max}(\ell^{max}) + \tilde{g}(r)$.

$$\left.\begin{array}{l} L^{max} + Rm \longrightarrow Q \\ \ell^{max} + rm \longmapsto f^{max}(\ell^{max}) + \tilde{g}(r) \end{array}\right\} \text{maybe clearer}$$

<u>Claim</u> F' is well-defined

pf.

If $\ell_1^{max} + r_1 m = \ell_2^{max} + r_2 m$,

then $(r_1 - r_2)m \overset{*}{=} \ell_2^{max} - \ell_1^{max} \in L^{max}$

$\Rightarrow r_1 - r_2 \in I$ (by the defn of I)

$$\Rightarrow \tilde{g}(\underbrace{r_1 - r_2}_{\in I}) = g(\underbrace{r_1 - r_2}_{\in I}) \underset{\text{by defn of } g}{=} f^{max}((r_1 - r_2)m) \underset{\text{by } *}{=} f^{max}(\ell_2^{max} - \ell_1^{max})$$

Therefore $\underbrace{f^{max}(\ell_1^{max}) + \tilde{g}(r_1)}_{= F'(\ell_1^{max} + r_1 m)} = \underbrace{f^{max}(\ell_2^{max}) + \tilde{g}(r_2)}_{= F'(\ell_2^{max} + r_2 m)}$ //

(Check this is indeed an R-map, by inspection)

Note that F' extends f to $L^{max} + Rm$:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, hook] \arrow[d, "f"'] & L^{max} \arrow[dl, "f^{max}"] \\ & Q & \end{tikzcd} \implies \begin{tikzcd} 0 \arrow[r] & L \arrow[r, hook] \arrow[d, "f"'] & \overbrace{L^{max} + Rm}^{M'} \arrow[dl, "F'"'] \\ & Q & \end{tikzcd}$$

$l^{max} + rm$

$\tilde{f}(l^{max}) + \tilde{g}(r)$

This contradicts maximality of L^{max}! $\quad$ ($m \notin L^{max}$ but $m \in L^{max} + Rm$)

Therefore $L^{max} = M$, and we can choose $\tilde{f} = f^{max}$.

Take a breather before we move on to ②, ③, + more, which are really just corollaries and much less intense.

Pf of 2

Assume R is a PID.

Every nonzero ideal $I \subset R$ is of the form $I = (r)$ $(r \neq 0)$.
(and conversely, $\forall\, r \neq 0$, (r) is an ideal.)

Then any R-map $f: I \longrightarrow Q$ is determined by
choice of $f(r) = q$.

This homomorphism can be extended to $\tilde{f}: R \longrightarrow Q$
iff $\exists\, q' \in Q$ such that $\tilde{f}(1) = q'$ and

$$q = f(r) = \tilde{f}(r) = r \cdot q'$$

(by R-linearity of $\tilde{f}$)

i.e. we need q' such that $q = rq'$. (and q was arbitrary, and there is a hom f for every choice $q \in Q$, $q = f(r)$!)

Baer's Criterion is therefore satisfied iff $rQ = Q$.

Pf. of 3

Suppose Q is an inj module over a PID,
ie $Q = rQ$ $\forall\, r \neq 0$ in R.

In a quotient module $\bar{Q} = Q/K$ where $K \subset Q$,
$\bar{Q} = r\bar{Q}$ still:

If $q_1 = rq_2$ then $\bar{q}_1 = r\bar{q}_2$ ... $\pi: Q \longrightarrow Q/K$ is an R-map!

Important Corollary to the Prop:

<u>Cor.</u> Every $\mathbb{Z}$-mod is a submodule of an injective $\mathbb{Z}$-mod.

(will need this to show every R-mod is a submodule of an injective R-mod).

Pf. Let $M \in \mathbb{Z}$-mod, B = set of $\mathbb{Z}$-mod generators for M.

Let $\mathcal{F} = F(B)$ = free $\mathbb{Z}$-mod on B.

$\Rightarrow$ SES $\quad 0 \to K \to \mathcal{F} \to M \to 0 \quad$ (identify $M = \mathcal{F}/K$)

Let $\mathcal{Q}$ ← \mathcal{Q} be the free $\mathbb{Q}$-mod on B.

$\Rightarrow \mathcal{Q} \cong \bigoplus_{i \in I} \mathbb{Q} \Rightarrow \mathcal{Q}$ is divisible $\Rightarrow \mathcal{Q}$ is injective.

Note that $\mathcal{Q}$ contains $\mathcal{F}$, which in turn contains K.

$\Rightarrow K$ is also a $\mathbb{Z}$-submodule of $\mathcal{Q} \Rightarrow \mathcal{Q}/K$ is injective (by ③ in prop)

$\Rightarrow M = \mathcal{F}/K \subseteq \mathcal{Q}/K$ where $\mathcal{Q}/K$ is injective.

HW 04: Prove this for general R-modules.

Compare this to: Every R-mod is a quotient of a projective (in fact free) R-mod.