Lecture 8

Today:

- finish injective modules
- more U.P. constructions: pullback, pushout, limit / colimit.

Recall

A $\mathbb{Z}$-mod A is divisible iff $A = nA$ $\forall n \neq 0$.

Last time we stated and discussed:

Prop. Let $Q \in {}_R\mathrm{Mod}$.

① (Baer's Criterion) Q is injective iff

$\forall$ left ideal $I \subset R$,

any R-hom $g: I \to Q$ can be extended to

an R-hom $\tilde{g}: R \to Q$.

← change of notation from previous statement of theorem (now my notation)

Recall left ideals $I \in {}_R\mathrm{Mod}$.

② If R is a PID, then Q is injective iff

$rQ = Q$ for every nonzero $r \in R$.

↳ In particular, a $\mathbb{Z}$-module $A \in \mathrm{Ab}$ is injective iff it is divisible, ie $A = nA$ $\forall n \in \mathbb{Z}$ where $n \neq 0$

ie divisible by all reasonable n

③ When R is a PID, quotient modules of injective R-mods are also injective.

Pf of ②

Assume R is a PID.

Every nonzero ideal $I \subset R$ is of the form $I = (r)$ $(r \neq 0)$.
(and conversely, $\forall\, r \neq 0$, (r) is an ideal.)

Then any R-map $f: I \longrightarrow Q$ is determined by choice of $f(r) = q$.

This homomorphism can be extended to $\tilde{f}: R \longrightarrow Q$
iff $\exists\, q' \in Q$ such that $\tilde{f}(1) = q'$ and

$$q = f(r) = \tilde{f}(r) = r \cdot q'$$

(by R-linearity of $\tilde{f}$)

ie. we need q' such that $q = rq'$.

(and q was arbitrary, and there is a hom f for every choice $q \in Q$, $q = f(r)$!)

Baer's criterion is therefore satisfied iff $rQ = Q$.

Pf. of ③

Suppose Q is an inj module over a PID,
ie $Q = rQ$ $\forall\, r \neq 0$ in R.

In a quotient module $\bar{Q} = Q/K$ where $K \subset Q$,
$\bar{Q} = r\bar{Q}$ still:

If $q_1 = rq_2$ then $\bar{q}_1 = r\bar{q}_2$ ... $\pi: Q \longrightarrow Q/K$ is an R-map!

Important Corollary to the Prop:

<u>Cor.</u> Every $\mathbb{Z}$-mod is a submodule of an injective $\mathbb{Z}$-mod.

(will need this to show every R-mod is a submodule of an injective R-mod).

Pf.

Let $M \in \mathbb{Z}$-mod, $B =$ set of $\mathbb{Z}$-mod generators for M.

Let $\mathcal{F} = F(B)$ = free $\mathbb{Z}$-mod on B.

$\Rightarrow$ SES $\quad 0 \to K \longrightarrow \mathcal{F} \longrightarrow M \longrightarrow 0 \quad$ (identify $M = \mathcal{F}/K$)

Let $Q \leftarrow$ \mathcal{Q} be the free $\mathbb{Q}$-mod on B.

$\Rightarrow Q \cong \bigoplus_{i \in I} \mathbb{Q} \Rightarrow Q$ is divisible $\Rightarrow Q$ is injective.

Note that Q contains $\mathcal{F}$, which in turn contains K.

$\Rightarrow$ K is also a $\mathbb{Z}$-submodule of $Q \Rightarrow Q/K$ is injective (by ③ in prop)

$\Rightarrow M = \mathcal{F}/K \subseteq Q/K$ where Q/K is injective.

HW 04: Prove this for general R-modules.

Compare this to: Every R-mod is a quotient of a projective (in fact free) R-mod.

Another pair of dual constructions via universal property:

Pullbacks

Let $\mathcal{C}$ be a category.

<u>defn.</u> Given $f: A \to C$ and $g: B \to C$ in $\mathcal{C}$,

a <u>solution</u> is an ordered triple (I would say, "the data")

(S, α, β) making the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "\beta"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

The "best" solution is called the <u>pullback</u>: for all solutions (X, α', β')...

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!\,\theta"] \arrow[drr, bend left, "\beta'"] \arrow[ddr, bend right, "\alpha'"'] & & \\ & S \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

This part should look familiar!

The pullback is also known as the <u>fiber product</u>

because of the pullback in Sets (and many categories

are built from Sets and Set morphisms:

eg. In Sets, suppose we are given

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Define $A \times_C B = \{(a,b) \in A \times B \mid f(a) = g(b)\}$

$$= \bigcup_{c \in f(A) \cap g(B)} \underbrace{f^{-1}(c) \times g^{-1}(c)}_{\text{cartesian product of fibers of the maps } f \text{ and } g}$$

with the induced projection maps (from $A \times B$)

$\alpha((a,b)) = a$ and $\beta(a,b) = b$.

ex. Check this satisfies the UP of pullback.

Rmk. ① pullbacks are unique up to unique isom

(again because they are defined by this kind of U.P. where $\exists! \theta$ ...)

② We sometimes write the pullback diagram like this

$$\begin{tikzcd} S \arrow[r, "\beta"] \arrow[d, "\alpha"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

to indicate that the "$\lrcorner$" part of the diagram was given to us and the other parts were filled in.

Pushouts

(I probably said pushforwards in class — that's something else at the morphism level!)

Make the dual definition: (Abridged here)

<u>defn</u> Given
$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \\ A & \end{tikzcd},$$

a <u>solution</u> is a triple (S, α, β) where

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & S \end{tikzcd}$$

commutes.

The "best solution" is the <u>pushout</u>: for all solutions (X, α', β')...

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "\beta"] \arrow[ddr, bend left, "\beta'"] & \\ A \arrow[r, "\alpha"] \arrow[drr, bend right, "\alpha'"'] & S \arrow[dr, dashed, "\exists!\,\theta"'] & \\ & & X \end{tikzcd}$$

The pushout is also sometimes called the fiber sum again because of our favorite category Sets:

eg. In Sets,
$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & A \amalg B / f(c) \sim g(c) \end{tikzcd}$$

α, β are induced by inclusion maps i_A, i_B into $A \amalg B$.

Rmk

① We sometimes draw the pushout diagram

as
$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & S \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

although some people will try to trick you and write

$$\begin{tikzcd} S & B \arrow[l] \\ A \arrow[u] & C \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

but as always, maps flow downward for me as much as possible

More examples:

① In ${}_R\text{Mod}$, $\ker\varphi$ is a pullback:

$$\begin{tikzcd} \ker\varphi \arrow[r, "0"] \arrow[d, "i"'] & 0 \arrow[d, "0"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

and $\operatorname{coker}\varphi$ is a pushout.

② In Sets,

if we start with inclusions, the pullback is $\cap$.

$$\begin{tikzcd} A\cap B \arrow[r] \arrow[d] & B \arrow[d, hook, "i_B"] \\ A \arrow[r, hook, "i_A"'] & C \end{tikzcd}$$

if we start with inclusions, the pushout is $\cup$:

$$\begin{tikzcd} C \arrow[r, hook, "j"] \arrow[d, hook, "i"'] & B \arrow[d, hook] \\ A \arrow[r, hook] & A\cup B \end{tikzcd}$$