

# Lecture 8

Today:

- finish injective modules
- more U.P. constructions: pullback, pushout, limit / colimit.

## Recall

A  $\mathbb{Z}$ -mod  $A$  is divisible iff  $A = nA \quad \forall n \neq 0$ .

Last time we stated and discussed:

prop. Let  $Q \in {}_R\text{Mod}$ .

① (Baer's Criterion)  $Q$  is injective iff

$\forall$  left ideal  $I \subset R$ ,

any  $R$ -hom  $g: I \rightarrow Q$  can be extended to

an  $R$ -hom  $\tilde{g}: R \rightarrow Q$ .

} Recall left ideals  
 $I \in {}_R\text{Mod}$ .

← change of notation for previous statement of theorem (now my notation)

② If  $R$  is a PID, then  $Q$  is injective iff

$rQ = Q$  for every nonzero  $r \in R$ .

↳ In particular, a  $\mathbb{Z}$ -module  $A \in \text{Ab}$  is injective iff

it is divisible, i.e.  $A = nA \quad \forall n \in \mathbb{Z}$  where  $n \neq 0$

↑ i.e. divisible by all reasonable  $n$

③ When  $R$  is a PID, quotient modules of injective  $R$ -mods are also injective.

Pf of ②

Assume  $R$  is a PID.

Every nonzero ideal  $I \subset R$  is of the form  $I = (r)$  ( $r \neq 0$ ).  
(and conversely,  $\forall r \neq 0$ ,  $(r)$  is an ideal.)

Then any  $R$ -map  $f: I \rightarrow Q$  is determined by  
choice of  $f(r) = q$ .

This homomorphism can be extended to  $\tilde{f}: R \rightarrow Q$

iff  $\exists q' \in Q$  such that  $\tilde{f}(1) = q'$  and

$$q = f(r) = \tilde{f}(r) = r \cdot q'$$

(by  $R$ -linearity of  $\tilde{f}$ )

i.e. we need  $q'$  such that  $q = r q'$ .

(and  $q$  was arbitrary,  
and there is a hom  $\tilde{f}$  for  
every choice  $q \in Q$ ,  $q = \tilde{f}(r)$ !)

Basis criterion is therefore satisfied iff  $rQ = Q$ .

Pf. of ③

Suppose  $Q$  is an inj module over a PID,

i.e.  $Q = rQ \forall r \neq 0$  in  $R$ .

In a quotient module  $\bar{Q} = Q/K$  where  $K \subset Q$ ,

$\bar{Q} = r\bar{Q}$  still:

If  $q_1 = r q_2$  then  $\bar{q}_1 = r \bar{q}_2 \dots$   $\pi: Q \rightarrow Q/K$  is an  $R$ -isompl.

Important Corollary to the Prop:

Cor. Every  $\mathbb{Z}$ -mod is a submodule of an injective  $\mathbb{Z}$ -mod.

(Will need this to show every  $R$ -mod is a submodule of an injective  $R$ -mod).

Pf.

Let  $M \in \mathbb{Z}$ -mod,  $B =$  set of  $\mathbb{Z}$ -mod generators for  $M$ .

Let  $\mathcal{F} = F(B) =$  free  $\mathbb{Z}$ -mod on  $B$ .

$\Rightarrow$  SES  $0 \rightarrow K \rightarrow \mathcal{F} \rightarrow M \rightarrow 0$  (identify  $M = \mathcal{F}/K$ )

Let  $\mathcal{Q} \leftarrow \{\text{mathcal{Q}}\} \mathbb{Q}$  be the free  $\mathbb{Q}$ -mod on  $B$ .

$\Rightarrow \mathcal{Q} \cong \bigoplus_{i \in I} \mathbb{Q} \Rightarrow \mathcal{Q}$  is divisible  $\Rightarrow \mathcal{Q}$  is injective.

Note that  $\mathcal{Q}$  contains  $\mathcal{F}$ , which in turn contains  $K$ .

$\Rightarrow K$  is also a  $\mathbb{Z}$ -submodule of  $\mathcal{Q} \Rightarrow \mathcal{Q}/K$  is injective (by ③ in prop)

$\Rightarrow M = \mathcal{F}/K \subseteq \mathcal{Q}/K$  where  $\mathcal{Q}/K$  is injective.  $\square$

HW04: Prove this for general  $R$ -modules.

Compare this to: Every  $R$ -mod is a quotient of a projective

(in fact free)  $R$ -mod.

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Another pair of dual constructions via universal property:

## Pullbacks

Let  $\mathcal{C}$  be a category.

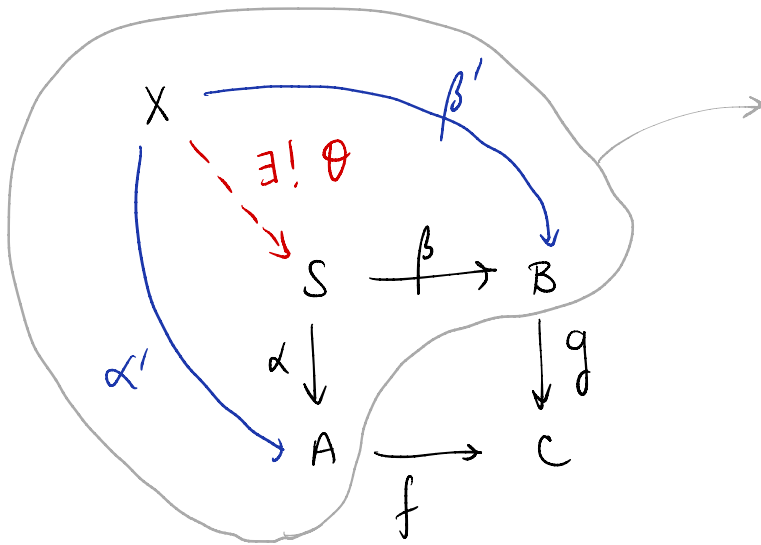
defn. Given  $f: A \rightarrow C$  and  $g: B \rightarrow C$  in  $\mathcal{C}$ ,

a solution is an ordered triple (I would say, "the data")

$(S, \alpha, \beta)$  making the following diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{\beta} & B \\ \alpha \downarrow & \circlearrowleft & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

The "best" solution is called the pullback: for all solutions  $(X, \alpha', \beta')$ ...



This part should look familiar!

The pullback is also known as the fiber product

because of the pullback in Sets (and many categories are built from Sets and Set morphisms):

eg. In sets, suppose we are given

$$\begin{array}{ccc} & & B \\ & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

Define  $A \times_c B = \{(a, b) \in A \times B \mid f(a) = g(b)\}$

$$= \bigcup_{c \in f(A) \cap g(B)} \underbrace{f^{-1}(c) \times g^{-1}(c)}_{\substack{\text{cartesian} \\ \text{product} \\ \text{of fibres} \\ \text{of the maps} \\ f \text{ and } g}}$$

with the induced projection maps (from  $A \times B$ )

$$\alpha(a, b) = a \text{ and } \beta(a, b) = b.$$

ex. Check this satisfies the UP of pullback.

Remk. ① pullbacks are unique up to unique ism  
(again because they are defined by this  
kind of U.P. where  $\exists! \theta \dots$ )

② We sometimes write the pullback diagram  
like this

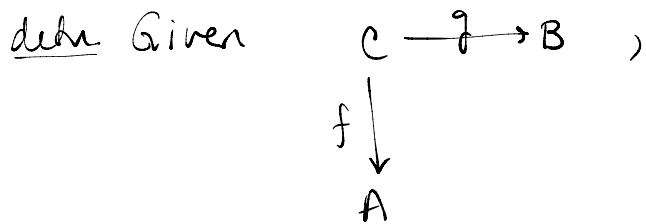
$$\begin{array}{ccc} S & \xrightarrow{\beta} & B \\ \alpha \downarrow & \lrcorner & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

to indicate that the " $\lrcorner$ " part of the diagram was  
given to us and the other parts were filled in.

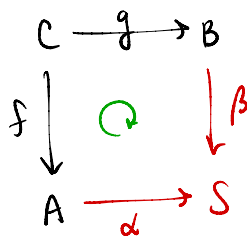
# Pushouts

(I probably said pushforwards in class - that's something else at the morphism level!)

Make the dual definition: (Abridged here)

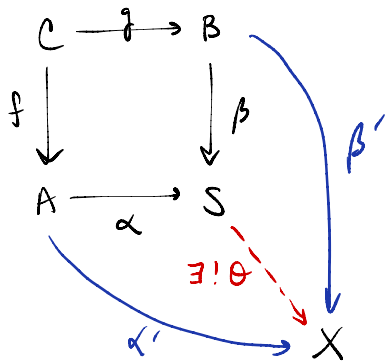


a solution is a triple  $(S, \alpha, \beta)$  where



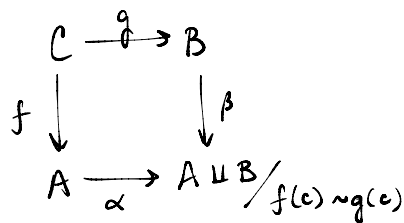
commutes.

The "best solution" is the pushout: for all solutions  $(X, \alpha', \beta')$ ...



The pushout is also sometimes called the fiber sum again because of our favorite category Sets:

eg. In Sets,



$\alpha, \beta$  are induced by inclusion maps  $i_A, i_B$  into  $A \sqcup B$ .

## Rank

① We sometimes draw the pushout diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ f \downarrow & & \downarrow \beta \\ A & \xrightarrow{\alpha} & S \end{array}$$

although some people will try to trick you and write

$$\begin{array}{ccc} S & \longleftarrow & B \\ \uparrow & & \uparrow g \\ A & \xleftarrow{f} & C \end{array}$$

but as always, maps flow downward for me as much as possible

## More examples:

① In  $\mathcal{R}\text{Mod}$ ,  $\ker \varphi$  is a pullback:

$$\begin{array}{ccc} \ker \varphi & \xrightarrow{\sigma} & 0 \\ i \downarrow & & \downarrow 0 \\ A & \xrightarrow{f} & C \end{array}$$

and  $\text{coker } \varphi$  is a pushout.

② In Sets,

if we start with inclusions, the pullback is  $\cap$ .

$$\begin{array}{ccc} A \cap B & \xrightarrow{\quad} & B \\ \downarrow & & \downarrow i_B \\ A & \xrightarrow{i_A} & C \end{array}$$

if we start with inclusions, the pushout is  $\cup$ :

$$\begin{array}{ccc} C & \xrightarrow{f} & B \\ i \downarrow & & \downarrow \\ A & \xrightarrow{\quad} & A \cup B \end{array}$$