

lecture 9

- tensor products of modules.

Tensor Product of modules over R

We will now use both left and right actions together.

Let R be a ring. For $A_R \in \text{Mod}_R$ and ${}_R B \in {}_R \text{Mod}$, we wish to understand their tensor product, written $M \otimes_R N$.

I will describe this in 2 ways:

- By univ. property (precise definition)
- How I actually think about it

defn. Let $A_R \in \text{Mod}_R$, ${}_R B \in {}_R \text{Mod}$.

Let G be an abelian group (additive). (Recall any module over any R is an abelian group)

A function $f: A \times B \rightarrow G$ is R -biadditive if

$$\forall a, a' \in A; b, b' \in B; r, r' \in R,$$

- $f(a+a', b) = f(a, b) + f(a', b)$
- $f(a, b+b') = f(a, b) + f(a, b')$
- $f(a, r b) = f(ar, b)$ (note action location)

Rmk An R -biadditive function is also called a pairing,

b/c you are kind of evaluating the pair (a, b)

in an r -linear way

eg. $\mu: R \times R \rightarrow R$ is R -biadditive:

Q. Why is $\mu(ar, b) = \mu(a, rb)$?

defn. If R is commutative, then for $A, B, M \in R\text{-mod}$,

a function $f: A \times B \rightarrow M$ is R -bilinear if

- f is biadditive and
- $f(ar, b) = f(a, rb) = r \cdot f(a, b)$

note Bilinearity is a property of f's like this in general

But later we will focus on bilinear module maps

eg. Let k be a field and $V \in \text{Vect}_k$, $V^* = \text{dual } V$.

evaluation $ev: V \times V^* \rightarrow k$ is k -bilinear:

recall $V^* = \text{Hom}_k(V, k)$

$$ev: V \times V^* \rightarrow k$$

$$(v, \varphi: V \rightarrow k) \mapsto \varphi(v)$$

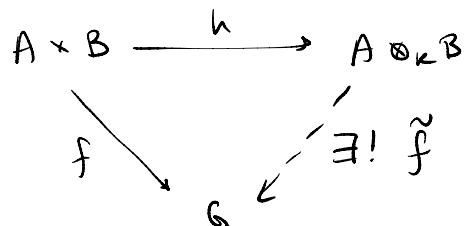
defn. Given R , $A_R \in \text{Mod}_R$, ${}_R B \in {}_R \text{Mod}$,

their tensor product is an abelian group $A \otimes_R B$

together with an R -biadditive fn. $h: A \times B \rightarrow A \otimes_R B$

such that $\forall G \in \text{Ab}$, $\forall R$ -biadditive $f: A \times B \rightarrow G$,

$\exists!$ \mathbb{Z} -map $\tilde{f}: A \otimes_R B \rightarrow G$ st.



Another UP.

\Rightarrow if exists, then
 unique up to
 unique isom

prop. Tensor products over R exist.

pf. (might be the construction you've seen before)

Candidate

Let $F =$ free \mathbb{Z} -mod gen'd by elements of $A \times B$.

\hookrightarrow ie write
 additively

Let $S =$ subgroup generated by the relations

$$(a, b+b') \sim (a,b) + (a,b')$$

$$(a+a', b) \sim (a,b) + (a',b)$$

$$(ar, b) \sim (a, rb)$$

Define $A \otimes_R B = F/S$.

$a \otimes b$ is the coset $(a,b) + S$

$$\begin{aligned}
 h: A \times B &\rightarrow A \otimes_R B \\
 (a,b) &\mapsto a \otimes b
 \end{aligned}$$

(just a restriction of
 $\pi: F \rightarrow F/S$)

Then

$$(a, b+b') \sim (ab) + (a, b') \implies a \otimes (b+b') = a \otimes b + a \otimes b'$$

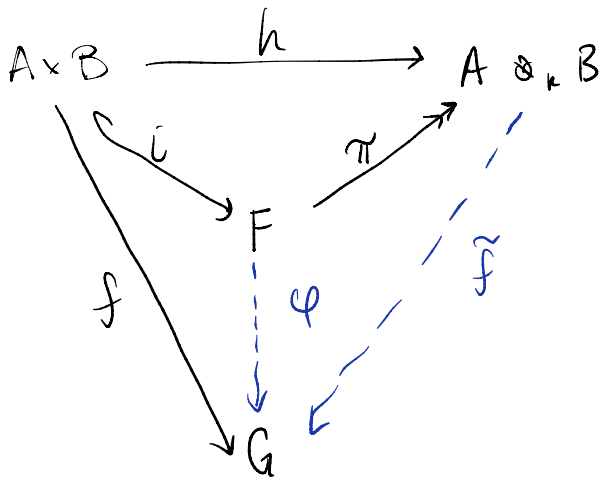
$$(a+a', b) \sim (a, b) + (a', b)$$

$$(ar, b) \sim (a, rb)$$

ETC

\implies clear that h is R -biadditive.

Now the UP:



$$\varphi: F \longrightarrow G$$

$$(a, b) \longmapsto f(a, b)$$

& extend \mathbb{Z} -linearly.

Observe $S \subseteq \ker \varphi$ as f is biadditive.

$$\implies \varphi \text{ induces } \tilde{f}: A \otimes_R B \longrightarrow G$$

$\underbrace{\hspace{1.5cm}}_{F/S}$

$$\text{where } \tilde{f}(a \otimes b) = \tilde{f}((a, b) + S) = \varphi(a, b) = f(a, b).$$

$$\implies \tilde{f}h = f \text{ as desired.}$$

Why \tilde{f} unique? $A \otimes_R B$ is generated by the pure tensors $\{a \otimes b\}$.

□

Remark $\forall u \in A \otimes_R B$, u can be written as

$$u = \sum_i a_i \otimes b_i \quad \text{non-uniquely.}$$

↑ note finite sums! bc that's what a free \mathbb{Z} -mod is in the first place.

How I actually work w/ tensor products:

$$A \otimes_R B = \langle \{a \otimes b\} \rangle$$

where adding is componentwise,

and r is small and can flow b/w the two sides.

Maps next time. in more detail.

eg. $R \otimes_R M \cong M$ why? (identify $r \otimes m$ w/ $1 \otimes rm$.)

eg. Künneth formula.

eg. $V^* \otimes V \cong \text{Hom}_k(V, V)$ \rightsquigarrow later: tensor-hom adjunction