Lecture 9

- tensor products of modules.

Tensor Product of modules over R

We will now use both left and right actions together.

Let R be a ring. For $A_R \in \mathrm{Mod}_R$ and ${}_RB \in {}_R\mathrm{Mod}$, we wish to understand their <u>tensor product</u>, written $M \otimes_R N$.

I will describe this in 2 ways:

- By univ. property (precise definition)
- How I actually think about it

<u>defn.</u> Let $A_R \in \mathrm{Mod}_R$, ${}_RB \in {}_R\mathrm{Mod}$.

Let G be an abelian group (additive). (Recall any module over any R is an abelian gp)

A function $f: A \times B \to G$ is <u>R-biadditive</u> if

$\forall\, a, a' \in A;\ b, b' \in B;\ r, r' \in R$,

- $f(a+a', b) = f(a,b) + f(a', b)$
- $f(a, b+b') = f(a,b) + f(a, b')$
- $f(a, rb) = f(ar, b)$ (note action location)

<u>Rmk</u> An R-biadditive function is also called a <u>pairing</u>, b/c you are kind of evaluating the pair (a,b) in an r-linear way

eg. $\mu: R \times R \to R$ is R-biadditive:

Q. Why is $\mu(ar, b) = \mu(a, rb)$?

defn. If R is commutative, then for $A, B, M \in R\text{-mod}$, a function $f : A \times B \longrightarrow M$ is <u>R-bilinear</u> if

- f is biadditive and
- $f(ar, b) = f(a, rb) = r \cdot f(a, b)$

note Bilinearity is a property of fns like this in general
But later we will focus on bilinear module maps

eg. Let k be a field and $V \in \mathrm{Vect}_k$, $V^* =$ dual VS.
evaluation $ev : V \times V^* \longrightarrow k$ is k-bilinear:

recall $V^* = \mathrm{Hom}_k(V, k)$

$$ev : V \times V^* \longrightarrow k$$
$$(v, \varphi : V \to k) \longmapsto \varphi(v)$$

defn. Given R, $A_R \in \mathrm{Mod}_R$, ${}_R B \in {}_R\mathrm{Mod}$,

their tensor product is an abelian group $A \otimes_R B$

together with an R-biadditive fn. $h : A \times B \longrightarrow A \otimes_R B$

such that $\forall\, G \in Ab$, $\forall$ R-biadditive $f : A \times B \longrightarrow G$,

$\exists!$ $\mathbb{Z}$-map $\tilde{f} : A \otimes_R B \to G$ s.t.

$$\begin{tikzcd} A \times B \arrow[r, "h"] \arrow[dr, "f"'] & A \otimes_R B \arrow[d, dashed, "\exists!\, \tilde{f}"] \\ & G \end{tikzcd}$$

another UP.

$\Rightarrow$ if exists, then unique up to unique isom

prop. Tensor products over R exist.

pf. (might be the construction you've seen before)

Candidate

Let F = free $\mathbb{Z}$-mod gen'd by elements of $A \times B$.

↳ ie write additively

Let S = subgroup generated by the relations

$(a, b+b') \sim (a,b) + (a,b')$

$(a+a', b) \sim (a,b) + (a',b)$

$(ar, b) \sim (a, rb)$

Define $A \otimes_R B = F/S$.

$a \otimes b$ is the coset $(a,b) + S$

$h : A \times B \longrightarrow A \otimes_R B$
$(a,b) \longmapsto a \otimes b$

(just a restriction of $\pi : F \longrightarrow F/S$)

Then

$(a, b+b') \sim (a,b) + (a,b')$ $\Longrightarrow$ $a \otimes (b+b') = a\otimes b + a \otimes b'$

$(a+a', b) \sim (a,b) + (a',b)$

ETC

$(ar, b) \sim (a, rb)$

$\Longrightarrow$ clear that h is R-biadditive.

<u>Now the UP:</u>

$$\begin{tikzcd}
A \times B \arrow[rr, "h"] \arrow[dr, bend left, "i"] \arrow[ddr, "f"'] & & A \otimes_R B \arrow[ddl, dashed, "\tilde{f}"] \\
& F \arrow[ur, "\pi"] \arrow[d, dashed, "\varphi"] & \\
& G &
\end{tikzcd}$$

$\varphi : F \longrightarrow G$
$(a,b) \longmapsto f(a,b)$

& extend $\mathbb{Z}$-linearly.

<u>Observe</u> $S \subseteq \ker \varphi$ as f is biadditive.

$\Longrightarrow$ φ induces $\tilde{f} : \underset{F/S}{\overset{\shortparallel}{A \otimes_R B}} \longrightarrow G$.

where $\tilde{f}(a \otimes b) = \tilde{f}((a,b)+S) = \varphi(a,b) = f(a,b)$.

$\Longrightarrow$ $\tilde{f} h = f$ as desired.

Why $\tilde{f}$ unique? $A \otimes_R B$ is generated by the pure tensors $\{a \otimes b\}$.

<u>Rmk</u> $\forall\ u \in A \otimes_R B$, u can be written as

$$u = \sum_i a_i \otimes b_i \qquad \text{noncuniquely.}$$

<u>note</u> finite sums! b/c that's what a free $\mathbb{Z}$-mod is in the first place.

How I actually work w/ tensor products:

$A \otimes_R B = \langle \{a \otimes b\} \rangle$

where adding is componentwise,

and r is small and can flow b/w the two sides.

<u>Maps</u> next time. in more detail.

eg. $R \otimes_R M \cong M$ Why? (identify $r \otimes_R m$ w/ $1 \otimes rm$.)

eg. Künneth formula.

eg. $V^* \otimes V \cong \mathrm{Hom}_k(V, V)$ $\rightsquigarrow$ later: tensor-hom adjunction