Lecture 10

<u>Recall</u> $R \in \mathrm{Rings}$, $A_R \in {}_R\mathrm{Mod}$, ${}_RB \in \mathrm{Mod}_R$ $\rightsquigarrow$ $A \otimes_R B \in \mathrm{Ab}$.

<u>Note</u> If ${}_RA_R \in {}_R\mathrm{Mod}_R$ (R–R bimodules),

then indeed $A \otimes_R B \in {}_R\mathrm{Mod}$.

$\hookrightarrow$ can check details on your own

eg. $V, W \in \mathrm{Vect}_k \Rightarrow V \otimes W \in \mathrm{Vect}_k$ as well

Maps

prop. Let $f : A_R \longrightarrow A'_R$ (morphism in Mod_R)

$g : {}_RB \longrightarrow {}_RB'$ (—— in ${}_R\mathrm{Mod}$)

Then $\exists!$ $\mathbb{Z}$-map denoted $f \otimes g : A \otimes_R B \longrightarrow A' \otimes_R B'$

such that $f \otimes g : a \otimes b \longmapsto f(a) \otimes g(b)$.

Recall the pure tensors $\{a \otimes b \mid a \in A, b \in B\}$ generate $A \otimes_R B$.

Pf.

$$\begin{tikzcd} A \times B \arrow[r, "h"] \arrow[dr, "\varphi"'] & A \otimes B \arrow[d, dashed, "\exists!\, \tilde{\varphi}"] \\ & A' \otimes B' \end{tikzcd}$$

$(a,b) \longmapsto f(a) \otimes g(b)$

$\tilde{\varphi} \rightarrow$ call this map "$f \otimes g$"

ISTS φ is R-biadditive.

- additivity on each side is clear ✓
- $(ar, b) \overset{\varphi}{\longmapsto} f(ar) \otimes g(b)$

$$= f(a) r \otimes g(b) \qquad \text{b/c } f, g \text{ are } R\text{-linear.}$$

$$= f(a) \otimes r g(b)$$

$$= f(a) \otimes g(rb) \overset{\varphi}{\longleftarrow\!\shortmid} (a, rb)$$

$\boxed{\Xi}$

We can also compose these maps b/w tensor products:

Cor. Given

$$A \xrightarrow{f} A' \xrightarrow{f'} A'' \quad (\text{in } \mathrm{Mod}_R)$$

$$B \xrightarrow{g} B' \xrightarrow{g'} B'' \quad (\text{in } {}_R\mathrm{Mod})$$

we have $(f' \otimes g') \circ (f \otimes g) = f'f \otimes g'g$

Pf. Because of the uniqueness of the map lifting

$\varphi: \; a \otimes b \longmapsto f'f(a) \otimes g'g(b)$. ∎

(Recall)

defn. Let $\mathcal{C}, \mathcal{D}$ be categories.

A functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a function such that

- if $A \in \mathcal{C}$, then $F(A) \in \mathcal{D}$.
- If $f \in \underline{\mathcal{C}(A, A')}$, then $F(f) \in \mathcal{D}(F(A), F(A'))$

  Common notation for $\mathrm{Mor}_{\mathcal{C}}(A, A') = \mathrm{Hom}_{\mathcal{C}}(A, A')$

- F preserves composition of morphisms

  $F(gf) = F(g) F(f)$
- F preserves identity morphisms

  $F(id_A) = id_{F(A)}$

thm. Fix $A_R \in \mathrm{Mod}_R$.

There is an additive functor

$$(A \otimes -) : {}_R\mathrm{Mod} \longrightarrow \mathrm{Ab}.$$

$$\text{(Ob)} \qquad {}_RB \longmapsto A \otimes_R B$$

$$\text{(Mor)} \qquad [g : B \to B'] \longmapsto [\mathrm{id}_A \otimes g : A \otimes B \longrightarrow A \otimes B']$$

covariant functor!

Pf.

① $A \otimes -$ is a functor:

identities are preserved: $\mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A \otimes B}$

(by uniqueness in UP maps)

composition:

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "A\otimes -"'] \arrow[rr, bend left, "g'g"] & B' \arrow[r, "g'"] \arrow[d, "A\otimes -"] & B'' \arrow[d, "A\otimes -"] \\
A \otimes B \arrow[r, "\mathrm{id}_A \otimes g"] \arrow[rr, bend right, "\mathrm{id}_A \otimes g'g"'] & A \otimes B' \arrow[r, "\mathrm{id}_A \otimes g'"] & A \otimes B''
\end{tikzcd}$$

ok, by above corollary ✓

② Additive functor:

Abelian cat: Hom sets are abelian groups

$F = A \otimes -$

NTS $F(g+h) = F(g) + F(h)$:

$$id_A \otimes (g+h) = id_A \otimes g + id_A \otimes h$$

✓ since both sides send the generators

$$a \otimes b \longmapsto a \otimes g(b) + a \otimes h(b).$$

$\blacksquare$

Rmk. Same construction for $- \otimes_R B : \mathrm{Mod}_R \longrightarrow \mathrm{Ab}$.

Cor. If $f : M \xrightarrow{\cong} M'$ and $g : N \xrightarrow{\cong} N'$ are isoms, then so is $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$.

Pf. $f \otimes g = (f \otimes id_{N'})(id_M \otimes g)$. //

Rmk.

① $M \in {}_R\mathrm{Mod} \implies R \otimes_R M \cong M$
 $r \otimes m \longmapsto rm$

② $M \in \mathrm{Mod}_R \implies M \otimes_R R \cong M$

③ $A \otimes \left(\bigoplus_{i \in I} B_i\right) \cong \bigoplus_{i \in I} A \otimes B_i$

(Again, uniqueness of the map in the UP)

$$\begin{tikzcd}
A \times \left(\bigoplus_i B_i\right) \arrow[r] \arrow[d] & A \otimes \left(\bigoplus_i B_i\right) \arrow[dl, dashed, "\exists !"] \\
\bigoplus_i A \times B_i &
\end{tikzcd}$$

$(a, b_1 + b_2)$

$a \otimes (b_1 + b_2)$
$= a \otimes b_1 + a \otimes b_2$

$a \otimes b_1 + a \otimes b_2$

note: finite sum everywhere

④ If k is commutative, then $A \otimes_k B \cong B \otimes_k A$

Key: May regard $M \in k\text{Mod}$ as a $(k-k)$ bimodule by enforcing $am = ma$.

Then.
$$\begin{tikzcd} A \times B \arrow[r] \arrow[dr] & A \otimes B \arrow[d, dashed, "\exists!"] \\ & B \otimes A \end{tikzcd} \qquad a \otimes b \mapsto b \otimes a.$$

Next time: Back to the story of proj. injective, ... etc.

thm. $A \otimes -$ is a (covariant) right-exact functor.