

Lecture 10

Recall $R \in \text{Rings}$, $A_R \in {}_R \text{Mod}$, ${}_R B \in \text{Mod}_R \rightsquigarrow A \otimes_R B \in \text{Ab}$.

Note. If ${}_R A_R \in {}_R \text{Mod}_R$ (R - R bimodules),

then indeed $A \otimes_R B \in {}_R \text{Mod}$.

\hookrightarrow can check details on your own

eg. $V, W \in \text{Vect}_k \Rightarrow V \otimes W \in \text{Vect}_k$ as well

Maps

prop. Let $f: A_R \rightarrow A'_R$ (morphism in Mod_R)

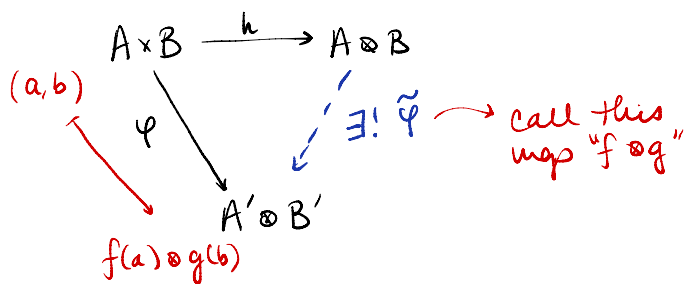
$g: {}_R B \rightarrow {}_R B'$ (— in ${}_R \text{Mod}$)

Then $\exists!$ \mathbb{Z} -map denoted $f \otimes g: A \otimes_R B \rightarrow A' \otimes_R B'$

such that $f \otimes g: a \otimes b \mapsto f(a) \otimes g(b)$.

Recall the pure tensors $\{a \otimes b \mid a \in A, b \in B\}$ generate $A \otimes_R B$.

Pf.



1STS φ is R -biadditive.

• additivity on each side is clear \checkmark

• $(ar, b) \xrightarrow{\varphi} f(ar) \otimes g(b)$

$$= f(a)r \otimes g(b)$$

$$= f(a) \otimes rg(b)$$

$$= f(a) \otimes g(rb) \xleftarrow{\varphi} (a, rb)$$

b/c f, g are R -linear.

\square

We can also compose these maps b/w tensor products:

Cor. Given $A \xrightarrow{f} A' \xrightarrow{f'} A''$ (in Mod_R)
 $B \xrightarrow{g} B' \xrightarrow{g'} B''$ (in ${}_R\text{Mod}$)

we have $(f' \otimes g') \circ (f \otimes g) = f'f \otimes g'g$

Pf. Because of the uniqueness of the map lifting

$$\varphi: a \otimes b \mapsto f'f(a) \otimes g'g(b). \quad \square$$

(Recall)

defn. Let \mathcal{C}, \mathcal{D} be categories.

A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a function such that

- if $A \in \mathcal{C}$, then $F(A) \in \mathcal{D}$.
- If $f \in \mathcal{C}(A, A')$, then $F(f) \in \mathcal{D}(F(A), F(A'))$

Common
notation for
 $\text{Mor}_{\mathcal{C}}(A, A') = \text{Hom}_{\mathcal{C}}(A, A')$

- F preserves composition of morphisms

$$F(gf) = F(g)F(f)$$

- F preserves identity morphisms

$$F(\text{id}_A) = \text{id}_{F(A)}$$

thm. Fix $A_R \in \text{Mod}_R$.

There is an additive functor

$$(A \otimes -): {}_R \text{Mod} \rightarrow \text{Ab}$$

$$(0_B) \quad {}_R B \mapsto A \otimes_R B$$

$$(Mor) \quad [g: B \rightarrow B'] \mapsto [id_A \otimes g: A \otimes B \rightarrow A \otimes B']$$

covariant functor!

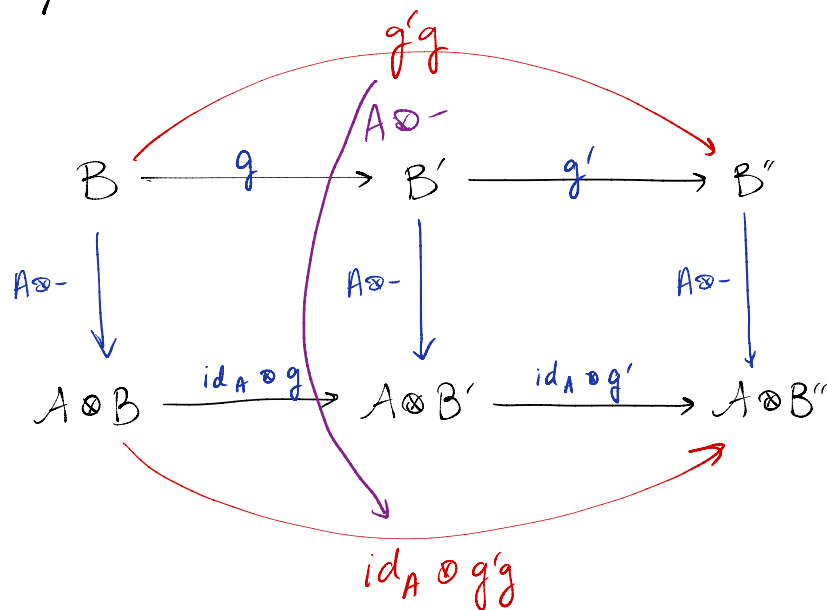
pf.

① $A \otimes -$ is a functor:

identities are preserved: $id_A \otimes id_B = id_{A \otimes B}$

(by uniqueness in UP maps)

composition:



ok, by above corollary ✓

② Additive functor:

Abelian cat: Hom sets are abelian groups

$$F = A \otimes -$$

$$\text{NTS } F(g+h) = F(g) + F(h):$$

$$\text{id}_A \otimes (g+h) = \text{id}_A \otimes g + \text{id}_A \otimes h$$

✓ since both sides send the generators

$$a \otimes b \mapsto a \otimes g(b) + a \otimes h(b).$$



Rule. Same construction for $- \otimes_R B: \text{Mod}_R \rightarrow \text{Ab}$.

Cor. If $f: M \xrightarrow{\cong} M'$ and $g: N \xrightarrow{\cong} N'$ are isoms, then so is $f \otimes g: M \otimes N \rightarrow M' \otimes N'$.

Pf. $f \otimes g = (f \otimes \text{id}_{N'}) (\text{id}_M \otimes g)$ //

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Rule

$$\textcircled{1} M \in {}_R \text{Mod} \Rightarrow R \otimes_R M \cong M$$

$r \otimes m \mapsto rm$

$$\textcircled{2} M \in \text{Mod}_R \Rightarrow M \otimes_R R \cong M$$

$$\textcircled{3} A \otimes \left(\bigoplus_{i \in I} B_i \right) \cong \bigoplus_{i \in I} A \otimes B_i$$

(Again, uniqueness of the map in the UP)

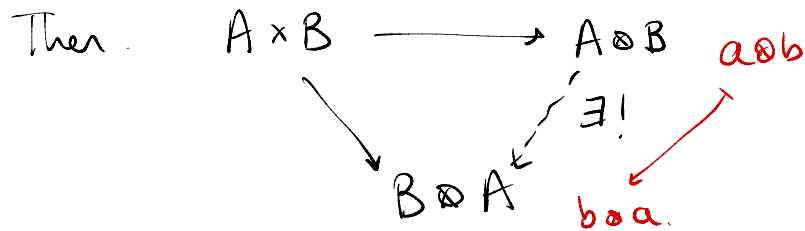
$$\begin{array}{ccc}
 A \times \left(\bigoplus_i B_i \right) & \longrightarrow & A \otimes \left(\bigoplus_i B_i \right) \\
 \downarrow & \swarrow \exists! & \downarrow \\
 \bigoplus_i A \times B_i & & a \otimes (b_1 + b_2) \\
 & & = a \otimes b_1 + a \otimes b_2
 \end{array}$$

$a \otimes b_1 + a \otimes b_2$

note: finite sum everywhere

④ If k is commutative, then $A \otimes_k B \cong B \otimes_k A$

Key: May regard $M \in k\text{-Mod}$ as a $(k-k)$ bimodule by enforcing $am = ma$.



Next time: Back to the story of proj. injective, ...
etc:

thm. $A \otimes -$ is a (covariant) right-exact functor.