Lecture 11

HW05 will be out TM...

<u>Recall</u> Let $A_R \in \mathrm{Mod}_R$. Then $A \otimes_R - : {}_R\mathrm{Mod} \longrightarrow \mathrm{Ab}$

is a covariant, additive functor

Ob: ${}_RB \longmapsto A \otimes_R B$

Mor: $f \longmapsto \mathrm{id} \otimes f$

$f+g \longmapsto \mathrm{id} \otimes (f+g) = \mathrm{id} \otimes f + \mathrm{id} \otimes g$

<u>thm.</u> $A \otimes_R -$ is right exact.

pf. Let $B \xrightarrow{i} C \xrightarrow{p} D \longrightarrow 0$ be an exact sequence of left R-modules.

WTS

$$A \otimes B \xrightarrow{\mathrm{id}_A \otimes i} A \otimes C \xrightarrow{\mathrm{id}_A \otimes p} A \otimes D \longrightarrow 0$$

is exact.

① $\mathrm{im}(\mathrm{id}_A \otimes i) \subseteq \ker(\mathrm{id}_A \otimes p)$

$(\mathrm{id}_A \otimes p)(\mathrm{id}_A \otimes i) = \mathrm{id}_A \otimes pi = \mathrm{id}_A \otimes 0 = 0.$

② $\ker(\mathrm{id}_A \otimes p) \subseteq \mathrm{im}(\mathrm{id}_A \otimes i)$ (pf. soon)

③ $\mathrm{id}_A \otimes p$ is surjective

Let $\sum a_i \otimes d_i \in A \otimes D$.

$\Rightarrow \exists\ c_i \in C$ s.t. $p(c_i) = d_i \ \forall i.$ (p is surjective)

Then $(\mathrm{id}_A \otimes p)\left(\sum a_i \otimes c_i\right) = \sum a_i \otimes d_i.$ ✓

② $\ker(id_A \otimes p) \subseteq \operatorname{im}(id_A \otimes i)$

let $I = \operatorname{im}(id_A \otimes i) \subset \ker(id_A \otimes p)$ by ①.

$$\begin{tikzcd}
 & I \subset \ker(id_A \otimes p) & & \\
A \otimes B \arrow[r, "id_A \otimes i"] & A \otimes C \arrow[u, phantom, "\cap"] \arrow[r, "id_A \otimes p"] & A \otimes D \arrow[r] & 0
\end{tikzcd}$$

So
$$\begin{tikzcd}
A \otimes C \arrow[r, "id \otimes p"] \arrow[d, "\pi"'] & A \otimes D \\
A \otimes C / I \arrow[ur, "\tilde{p}"'] &
\end{tikzcd}$$
factors through $A \otimes C / I$:

where $a \otimes c + I \longmapsto a \otimes p(c)$

Now we show $\tilde{p}$ is actually an isomorphism.

↪ in which case $\ker(id \otimes p) = I = \operatorname{img}(id \otimes i)$.

We will define an inverse to $\tilde{p}$, called $\tilde{q}$.

$$\begin{tikzcd}
 & I \subset \ker(id_A \otimes p) & & \\
A \otimes B \arrow[r, "id_A \otimes i"] & A \otimes C \arrow[u, phantom, "\cap"] \arrow[r, "id_A \otimes p"] \arrow[d, "\pi"'] & A \otimes D \arrow[r] \arrow[dl, dashed, bend left, "\tilde{q}"] & 0 \\
 & A \otimes C / I \arrow[ur, "\tilde{p}"] & & \\
 & & & A \times D \arrow[ull, "q"] \arrow[uul]
\end{tikzcd}$$

First define $q : \overset{\text{just product}}{A \times D} \longrightarrow A \otimes C/I$

Let $(a, d) \in A \times D$.

Then since p is surjective, $\exists\, c \in C$ such that $p(c) = d$.

Define $q((a,d)) = a \otimes c + I$.

Need to check q is well-defined!

If $p(c') = d$, then $p(c - c') = p(c) - p(c') = 0$

$\Rightarrow c - c' \in \ker p = \operatorname{im} i$.

$\Rightarrow \exists\, b \in B$ s.t. $i(b) = c - c'$.

$\Rightarrow a \otimes (c - c') = a \otimes i(b) \in \operatorname{im}(\mathrm{id}_A \otimes i) = I$. ✓

Check that q is R-biadditive: $q((a,d)) = a \otimes c + I$.

e.g. $(a, d_1 + d_2) \longmapsto a \otimes (c_1 + c_2) + I$ etc.

$(ar, d) \mapsto ar \otimes c + I = a \otimes rc + I$, and $p(rc) = rd$ indeed

$\Rightarrow$ By U.P. of $\otimes$, we obtain a map

$$\tilde{q} : A \otimes D \longrightarrow (A \otimes C)/I$$
$$a \otimes d \longmapsto a \otimes c + I \quad \text{where } p(c) = d.$$

Check that $\tilde{q}$ and $\tilde{p}$ are inverses.

(suffices to check on generators)

$$\begin{tikzcd}
a \otimes c + I \arrow[r, mapsto] & a \otimes p(c) \\
A \otimes C/I \arrow[r, shift left, "\tilde{p}"] & A \otimes D \arrow[l, shift left, "\tilde{q}"] \\
a \otimes c + I & a \otimes d \arrow[l, mapsto]
\end{tikzcd}$$

where $p(c) = d$.

Q. When is $A \otimes_R -$ also left exact?

i.e. for which $A_R \in \mathrm{Mod}_R$ is the covariant functor $A \otimes_R -$ exact?

defn. A right R-module A_R is flat if $A \otimes_R -$ is an exact functor. (${}_R B \in {}_R\mathrm{Mod}$ is flat if $- \otimes_R B$ is exact.)

Q. Are there any nonflat modules? When is $A \otimes -$ not exact?

eg. In $\mathbb{Z}$-mod, consider

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{p} \mathbb{Q}/\mathbb{Z} \to 0.$$

let $A = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$

Then after applying $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} -$, we have

$$0 \to \underbrace{\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}}_{\cong \mathbb{Z}/2\mathbb{Z}} \xrightarrow{id \otimes i} \underbrace{\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}} \xrightarrow{id \otimes p} \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \to 0$$

$$\begin{aligned} &\bar{1} \otimes r \\ &= \bar{1} \otimes 2 \cdot (\tfrac{1}{2} r) \\ &= \bar{1} \cdot 2 \otimes \tfrac{1}{2} r \\ &= 0 \otimes \tfrac{1}{2} r. \\ &\Rightarrow \cong 0 \quad (!!!) \end{aligned}$$

$\Rightarrow$ $id \otimes i$ is NOT injective.

(BTW) Prop. $\forall B \in \mathbb{Z}$-mod, $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} B \cong B/nB$

for the same reason (pf more cplx).

We will do more w/ tensor products next week.