Lecture 12

<u>Aside</u>: <u>Localization</u>

eg. Prototypical example: $R = \mathbb{Z}$. Let $D = \mathbb{Z} - \{0\}$.

D is multiplicatively closed: $1 \in D$, and $a, b \in D \Rightarrow ab \in D$.

"denominators"

$\Rightarrow D^{-1}\mathbb{Z} = \mathrm{Frac}(\mathbb{Z}) = \mathbb{Q}$.

In general, if R is a commutative ring,

and D is a multiplicatively closed subset of R,

then $D^{-1}R$ is the <u>ring of fractions of R wrt D</u>

or the <u>localization of R at D</u>.

what I always say

Explicitly, $D^{-1}R = R \times D/\sim$ where

$(r,d) \sim (s,e)$ iff $\exists\, x \in D$ such that

$x(er - ds) = 0$

Just think fractions: if all $r, d, s, e \in \mathbb{Z}$, then

$\frac{r}{d} \sim \frac{s}{e}$ iff $er - ds = 0$.

But if R isn't an integral domain, it may have zero divisors.

$D^{-1}R$ is a ring: add + multiply as you would w/ fractions.

Now you can also <u>localize an R-module at D</u>:

$M \in R\text{-mod} \rightsquigarrow D^{-1}R \otimes_R M$.

(Aside also?) <u>Tensor Product of R-algebras.</u>

Let k be a commutative ring.

Then a ring R is a k-algebra if

R is a k-module and

scalars in k commute with everything.

eg. • polynomial algebras: $k[X]$, $k[X]/X^n$

• or a noncommutative one:

$k[X,Y]$ where $YX = -XY$.

• $k[G]$

• every R is a $\mathbb{Z}$ algebra

• if $k \subset Z(R)$ (center) then R is a k-alg.

<u>Prop.</u> If $k \in$ CommRing, $A, B \in k$-alg, then

$A \otimes_k B$ is also a k-alg by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.

<u>prop</u> Let $R, S \in k$-alg.

Then every (R,S)-bimodule M is a left

$R \otimes_k S^{op}$-module, where $(r \otimes s)m = rms$.

Pf. Use notion of trilinearity in k ..

Tensor-Hom adjunction

<u>defn</u>. Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be covariant functors.

A <u>natural transformation</u> is a family of morphisms $\tau = (\tau_C : F(C) \longrightarrow G(C))_{C \in obj(\mathcal{C})}$ such that

this diagram commutes for all $f : C \longrightarrow C'$ in $\mathcal{C}$:

$$\begin{tikzcd} F(C) \arrow[r, "Ff"] \arrow[d, "\tau_C"] \arrow[dr, phantom, "\circlearrowleft"] & F(C') \arrow[d, "\tau_{C'}"] \\ G(C) \arrow[r, "Gf"] & G(C') \end{tikzcd}$$

If each τ_C is an isomorphism, then τ is a <u>natural isom</u>.

and F and G are <u>naturally isomorphic functors</u>.

eg. Let k be a field and let $V \in Vect_k = k\text{-Mod}$

Recall $V^* = Hom_k(V, k)$ is the dual VS.

The evaluation map $ev_v : f \longmapsto f(v)$ is a linear functional on V^*, ie

$$ev_w \in (V^*)^* = V^{**}.$$

Define $\tau_V : V \longrightarrow V^{**}$
$\qquad\qquad v \longmapsto ev_v$

① τ is a natural transformation from

$F = id$ functor on $Vect_k$ to

$G =$ double dual functor on $Vect_k$.

② When restricted to the subcategory of $FDVect_k$, τ is a natural isomorphism.

Two more examples

① prop. Recall $\varphi_M : R \otimes M \xrightarrow{\cong} M$

$$f \longmapsto f(1)$$

These maps $\varphi = (\varphi_M)_{M \in {}_R\mathrm{Mod}}$ form a natural isom $\mathrm{Hom}_R(R, -) \longrightarrow \mathrm{id}_{{}_R\mathrm{Mod}}$.

pf. Check the naturality diagram commutes:

Suppose $h \in \mathrm{Hom}_R(M, N)$.

$$\begin{tikzcd}
\mathrm{Hom}_R(R, M) \arrow[r, "h_*"] \arrow[d, "\varphi_M"'] & \mathrm{Hom}_R(R, N) \arrow[d, "\varphi_N"] \\
M \arrow[r, "h"'] & N
\end{tikzcd}$$

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② prop. Recall $\theta_M : R \otimes_R M \xrightarrow{\cong} M$

$$r \otimes m \longmapsto rm$$

These $\theta = (\theta_M)_{M \in {}_R\mathrm{Mod}}$ form a natural isom fcn. $(R \otimes_R -) \longrightarrow \mathrm{id}_{{}_R\mathrm{Mod}}$.

Pf. Again, say $h : M \to N$.

$$\begin{tikzcd}
R \otimes_R M \arrow[r, "1 \otimes h"] \arrow[d, "\theta_M"'] \arrow[dr, phantom, "\circlearrowleft"] & R \otimes_R N \arrow[d, "\theta_N"] \\
M \arrow[r, "h"] & N
\end{tikzcd}$$

Thm. Let R, S be rings.

Given modules A_R, ${}_R B_S$, ${}_S C$ in the appropriate categories.

There is an isomorphism of abelian groups

$$\tau_{A,B,C} : \mathrm{Hom}_S(A \otimes_R B, C) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)). \quad (*)$$

$$[f : A \otimes_R B \longrightarrow C] \longmapsto \left(f_a^* : B \longrightarrow C\right)_{a \in A}$$

$$b \longmapsto f(a \otimes b)$$

the following agrees w/ the above.

Indeed, if we fix two out of three in $\{A, B, C\}$, the maps $\tau_{A,B,C}$ constitute natural isomorphisms

① $\mathrm{Hom}_S(- \otimes_R B, C) \longrightarrow \mathrm{Hom}_R(-, \mathrm{Hom}_S(B, C))$

② $\mathrm{Hom}_S(A \otimes_R -, C) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_S(-, C))$

③ $\mathrm{Hom}_S(A \otimes_R B, -) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, -))$

Let's not prove this today; let's discuss the significance

If we define the functors

$$F = - \otimes_R B \quad \text{and} \quad G = \mathrm{Hom}_S(B, -),$$

$$F : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_S \qquad G : \mathrm{Mod}_S \longrightarrow \mathrm{Mod}_R.$$

then we can rewrite the above $(*)$ as

$$\tau : \mathrm{Hom}_S(FA, C) \longrightarrow \mathrm{Hom}_R(A, GC)$$