Lecture 14

Regarding Proof from last class:

? error in book?

Prop. Let R be a ~~commutative~~ ring. $A, B, C \in R\text{-mod}$.

$$\left(A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0 \qquad \text{is exact} \right)$$

if and only if

For every $X \in R\text{-mod}$,

$$0 \longrightarrow \mathrm{Hom}_R(C, M) \xrightarrow{p^*} \mathrm{Hom}_R(B, M) \xrightarrow{i^*} \mathrm{Hom}_R(A, M)$$

is exact.

Pf.

$\mathrm{Hom}_R(-, M)$ for each M is left exact, so ISTS the "$\Leftarrow$" direction.

Check ① p surj ② $\mathrm{im}(i) \subseteq \ker p$ ③ $\ker p \subseteq \mathrm{im}(i)$

by diagram chasing, essentially

see paper notes (If you want me to post these, let me know. Proof is also in the book.)

Last time:

Adjoint Isom: A_R, ${}_R B_S$, C_S (ie adjunction for right modules)

$$\tau_{A,B,C} : \mathrm{Hom}_S(A \otimes_R B, C) \xrightarrow{\cong} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \qquad \text{(in Ab)}$$

Adjoint Isom II: ${}_R A$, ${}_S B_R$, ${}_S C$ (ie adjunction for left modules)

$$\tau'_{A,B,C} : \mathrm{Hom}_S(B \otimes_R A, C) \xrightarrow{\cong} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \qquad \text{(in Ab)}$$

Similarly, we obtain natural isoms by fixing two out of $\{A, B, C\}$

In particular,

natural isomorphisms

$$\mathrm{Hom}_S(B \otimes_R -, C) \xrightarrow{\cong} \mathrm{Hom}_R(-, \mathrm{Hom}_S(B, C))$$

Flat Modules

<u>Recall</u> $A \otimes_R -$ is right exact (covariant).

$A \in \mathrm{Mod}_R$ is a <u>flat module</u> if $A \otimes_R -$ is exact.

$\Rightarrow$ $A \in \mathrm{Mod}_R$ iff $id_A \otimes -$ preserves injections.

i.e. $\ker(id_A \otimes i) = 0$ when i is an injection.

<u>Rmk</u> The deficit is measured by the Tor functor. $\mathrm{Tor}_1^R(A, -)$

Why the terminology? Tor ~ Torsion.

e.g. Torsion free abelian groups are flat $\mathbb{Z}$-modules HW06

(Cor. 8.7... If R is a PID, then $M \in R$-mod is flat iff tor-free)

(R Noetherian, or ID, $\Rightarrow$ f.g. flat $\Rightarrow$ projective.)

Similar defn for flat left-modules: $- \otimes_R B$ is exact.

<u>More examples</u>:

① $R \in {}_R\mathrm{Mod}$, Mod_R is flat

② $\bigoplus_j M_j$ is flat iff M_j is flat $\forall j$

③ Every projective right R-module F is flat.

Same statement for left R-modules.

But to keep things straight, lets focus on the functor $A \otimes -$ today.

Pf.

①

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "1\otimes -"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "1\otimes -"] \\ R\otimes_R A \arrow[r, "id_R\otimes i"] & R\otimes_R B \end{tikzcd}$$

} both are isoms.

$\Longrightarrow$ $id_R \otimes i$ also injective.

②

$$\begin{tikzcd} (\bigoplus_j M_j) \otimes A \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & (\bigoplus_j M_j) \otimes_R B \arrow[d, "\cong"] \\ \bigoplus_j (M_j \otimes A) \arrow[r] & \bigoplus_j (M_j \otimes_R B) \end{tikzcd}$$

(We didn't prove the $\cong$ in class)

③ by ① & ②, a free right module is flat.

Now a module is projective iff it's a summand of a free module, so by ② is also flat.

$\boxed{\text{projective} \Longrightarrow \text{flat}}$

Now the proof that $\mathrm{Hom}_{\mathbb{Z}}(R,D)$, D = divisible abelian gp is an injective left R-module: (cf. HW03 Ex 2c)

prop. If $B \in \mathrm{Mod}_R$ is flat and D is a divisible Abelian gp (i.e. injective $\mathbb{Z}$-mod) then $\mathrm{Hom}_{\mathbb{Z}}(B,D)$ is an injective left R-module.

e.g. let $B = R$, which we showed was flat as an R-module.

Pf. View $B \in \mathrm{Mod}_R$ as a $(\mathbb{Z}, R)$-bimodule.

$\Rightarrow \mathrm{Hom}_{\mathbb{Z}}(B,D)$ is a left R-module

(see 1st page of notes from last lecture)

To show $\mathrm{Hom}_{\mathbb{Z}}(B,D)$ is injective, ISTS

$\mathrm{Hom}_R(-, \mathrm{Hom}_{\mathbb{Z}}(B,D))$ is an exact functor

By Adjoint Isom (II), we have natural isomorphisms

$$\tau_A : \underbrace{\mathrm{Hom}_{\mathbb{Z}}(B \otimes_R A, D)}_{} \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(B, D))$$

Notice this functor is just the composition

$$A \longrightarrow \underbrace{B \otimes_R}_{\substack{B \text{ is flat} \\ \Rightarrow \text{functor is} \\ \text{exact}}} A \longrightarrow \underbrace{\mathrm{Hom}_{\mathbb{Z}}(B \otimes_R A, D)}_{\substack{D \text{ injective} \\ \Rightarrow \text{functor is exact}}}$$

Composition of exact functors is exact. $\Rightarrow \mathrm{Hom}_{\mathbb{Z}}(B \otimes_R -, D)$ is exact.

$\Rightarrow \mathrm{Hom}_R(-, \mathrm{Hom}_{\mathbb{Z}}(B,D))$ exact $\Rightarrow \mathrm{Hom}_{\mathbb{Z}}(B,D)$ is injective. ☒