# Lecture 15

- Wednesday: Midterm! Know your definitions and main thms!

Moving toward relating flat w/ injective, projective

But today, we'll discuss categorical <u>limits</u> and <u>colimits</u>

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Let $(I, \leq)$ be a partially ordered set. $\rightsquigarrow$ gives a category $PO(I)$

e.g. $(\mathbb{Z}, \leq) \rightsquigarrow \left( \cdots \to \underset{-1}{\bullet} \to \underset{0}{\bullet} \to \underset{1}{\bullet} \to \cdots \right)$

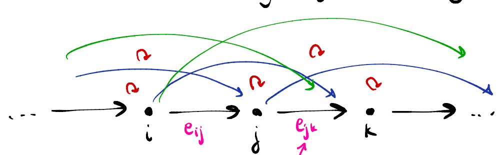
$\bullet$ objects

$\to$ morphisms

These will be the shapes of the diagrams we draw.

⚠ Recall, morphisms can be composed!

So the diagram category really looks like

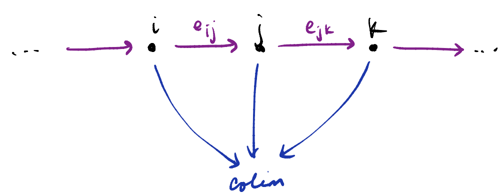



not standard notation; just means $i \leq j$, $j \leq k$

# Bob Ross Cartoons:

* Not precise definitions here (go read the/a book) but rather how to remember what data limits + colims are built from.
(UP defn later in lecture in context of modules).

## Direct limits aka. colimits aka. inductive limit. aka $\varinjlim$




direct system: if $i \le j \le k$, then

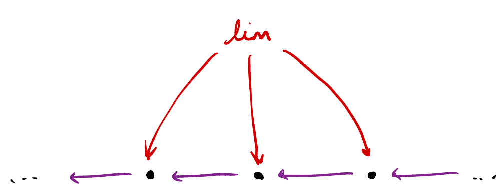
need

$$\begin{tikzcd} F_i \arrow[r, "F e_{ij}"] \arrow[rr, bend right, "F e_{ik}"'] & F_j \arrow[r, "F e_{jk}"] \arrow[d, phantom, "\circlearrowright"] & F_k \\ & {} & \end{tikzcd}$$

* The direction of the purple arrows is mathematically immaterial; it's just convention for the functor from PO(I)

* The direction of the BLUE arrows are what make this a colim.

## Inverse limits aka limits aka projective limits aka. $\varprojlim$




(same as above)

inverse system: if $k \geq j \geq i$

$$\begin{tikzcd} G_i & G_j \arrow[l, "G e^{ij}"'] \arrow[d, phantom, "\circlearrowright"] & G_k \arrow[l, "G e^{jk}"'] \arrow[ll, bend left, "G e^{ik}"] \\ & {} & \end{tikzcd}$$

I can't remember all these terms so I just use "limit" and "colim".

defn. $I$ = poset. Direct system of left $R$-mods over $I$ is an ordered pair denoted $\{M_i, \varphi_i^j\}$

- indexed family of modules $(M_i)_{i\in I}$
- morphisms: $(\varphi_j^i : M_i \longrightarrow M_j)$ where

$$\varphi_i^i = 1_{M_i}$$

and this diagram commutes whenever $i \preceq j \preceq k$:

$$\begin{tikzcd} M_i \arrow[r, "\varphi_j^i"] \arrow[rr, bend right, "\varphi_k^i"'] & M_j \arrow[r, "\varphi_k^j"] & M_k \end{tikzcd}$$

ie. This $\{M_i, \varphi_i^j\}$ is really the data of a covariant functor $F: PO(I) \longrightarrow {}_R\mathrm{Mod}$

defn $I$ = poset. Inverse system of left-$R$-mods over $I$ is an ordered pair denoted $\{M_i, \psi_i^j\}$

- indexed family of modules $(M_i)_{i\in I}$
- morphisms: $(\psi_i^j : M_j \longrightarrow M_i)$ where

$$\psi_i^i = 1_{M_i}$$

and this diagram commutes whenever $i \preceq j \preceq k$:

$$\begin{tikzcd} M_k \arrow[r, "\psi_i^k"] \arrow[rr, bend right, "\psi_i^k"'] & M_j \arrow[r, "\psi_i^j"] & M_i \end{tikzcd}$$

ie. This $\{M_i, \psi_i^j\}$ is really the data of a contravariant functor $F: PO(I) \longrightarrow {}_R\mathrm{Mod}$

Let's study <u>colimits</u> (you can read about the dual)

# Colimits = direct limits

<u>defn.</u> Let $I$ = poset, $\{M_i, \varphi_j^i\}$ a direct (or directed) system of left $R$-modules over $I$.

The <u>colimit</u> is a left-$R$-mod $\cancel{\varinjlim}\,\operatorname{colim} M_i$ along with a family of $R$-maps $(\alpha_i : M_i \longrightarrow \varinjlim M_i)_{i \in I}$ such that

- $\alpha_j \varphi_j^i$ whenever $i \leq j$

$$\begin{tikzcd} M_i \arrow[rr, "\varphi_j^i"] \arrow[dr, "\alpha_i"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & M_j \arrow[dl, "\alpha_j"] \\ & \operatorname{colim} \cancel{\varinjlim} M_k & \end{tikzcd}$$

- for any module $X$ w/ maps $(f_i : M_i \longrightarrow X)$ also satisfying $f_j \varphi_j^i = f_i$ $\forall\, i \leq j$,

  $\exists!$ $R$-map $\theta : \cancel{\varinjlim}\,\operatorname{colim} M_i \longrightarrow X$ making the diagram commute:

$$\begin{tikzcd} M_i \arrow[rr, "\varphi_j^i"] \arrow[dr, "\alpha_i"] \arrow[ddr, bend right, "f_i"'] & & M_j \arrow[dl, "\alpha_j"'] \arrow[ddl, bend left, "f_j"] \\ & \operatorname{colim} \cancel{\varinjlim} M_k \arrow[d, "\exists!\,\theta"] & \\ & X & \end{tikzcd}$$

## Examples

① coproduct

$$\begin{tikzcd}
M_i \arrow[r, "0"] \arrow[dr, "\alpha_i"'] & M_j \arrow[d, "\alpha_j"] \\
& \coprod M_k
\end{tikzcd}$$

② pushout

Q. What's the index category?

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & D \arrow[dr, "\exists!"] & \\
& & X
\end{tikzcd}$$

prop. $\mathrm{Hom}_R(\operatorname{colim} M_i, B) \cong \operatorname{colim} \mathrm{Hom}_R(M_i, B)$

i.e. $\mathrm{Hom}_R(-, B)$ preserves colimits.

pf. HW07, follow dual proof in book.

(Limits)

defn.

$$\begin{tikzcd} & \lim M_i \arrow[dl, "\alpha_i"'] \arrow[dr, "\alpha_j"] \arrow[d, phantom, "\circlearrowright"] & \\ M_i & & M_j \arrow[ll] \end{tikzcd}$$

Universal property:

$$\begin{tikzcd} & X \arrow[d, "\exists!\,\theta"] \arrow[ddl, bend right, "f_i"'] \arrow[ddr, bend left, "f_j"] & \\ & \lim M_i \arrow[dl, "\alpha_i"'] \arrow[dr, "\alpha_j"] & \\ M_i & & M_j \arrow[ll] \end{tikzcd}$$

Examples

(1) product

(2) pullback

(3) p-adic integers

prop. $\mathrm{Hom}_R(A, \lim M_i) \cong \lim \mathrm{Hom}_R(A, M_i)$

$\mathrm{Hom}_R(A, -)$ preserves limits.

pf. In the book.

Next time: Actual proofs. Relation w/ $\otimes$ and flat modules.