Lecture 17

Today: Bilinear forms

Topological motivations:

1. Intersection forms for ~~smooth~~ closed 4-manifolds
2. Seifert (linking matrix) form for links
3. Toward exterior algebras, differential forms (Friday)

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defn. A <u>bilinear form</u> or <u>inner product</u> on a finite dimensional vectorspace V over a field k is a bilinear function

$$f : V \times V \longrightarrow k \qquad (\Rightarrow \exists!\ \tilde{f} : V \otimes V \to k)$$

The pair (V, f) is called an <u>inner product space</u> over k.

eg. (k^n, dot product) $f(u,v) = \sum_i u_i v_i$

$$= u^T v = u^T \cdot I_n v$$

swap out this square matrix to get a different bilinear form.

<u>defn.</u> A bilinear form f is <u>symmetric</u> if
$f(u,v) = f(v,u)$ for all $u, v \in V$.

$\Rightarrow$ the matrix associated to f (after choosing a basis) is symmetric.

- (V, f) is called a "symmetric" space.

<u>defn.</u> A bilinear form f is <u>alternating</u> if $f(v,v) = 0$ for all $v \in V$.

- (V, f) is then called an "alternating" space

<u>defn.</u> f is <u>skew</u> if $f(v,u) = -f(u,v)$ for all $u, v \in V$.

<u>Rmk.</u> If char $k \neq 2$, then f is alternating iff f is skew.
(The matrix is skew-symmetric.)

<u>prop.</u> Every bilinear form f on V over k where <u>char $k \neq 2$</u> can be expressed <u>uniquely</u> in terms of symmetric + alternating forms.

pf.

$$f_s(u,v) = \frac{1}{2}\left(f(u,v) + f(v,u)\right)$$

$$f_a(u,v) = \frac{1}{2}\left(f(u,v) - f(v,u)\right).$$

If $f = f_s + f_a = f_s' + f_a'$, then

$g = f_s - f_s' = f_a' - f_a$ is simultaneously symmetric and alternating $\Rightarrow g = 0$. ∎

There are bilinear forms that are more interesting than others: <u>nondegenerate</u> forms. We will take this characterization as the definition (see book for other defns).

<u>defn</u> f is nondegenerate iff

$$f(u,v) = 0 \quad \forall\, v \in V \Rightarrow u = 0.$$

and $f(u,v) = 0 \quad \forall\, u \in V \Rightarrow v = 0$

<u>thm</u>. Let (V, f) be an inner product space. Let $e_1, \dots, e_n$ be a basis for V. Then f is nondegenerate iff the dual basis $\{e_i^* = f(-, e_i)\}$ is a basis of the dual space V^*.

Pf.

① (If f is nondegenerate).
Suffices to show linear independence since $\dim V^* = \dim V$.

If $\exists \{c_i\}$ s.t. $\sum_i c_i e_i^* = 0$

then $\underbrace{\sum_i c_i f(v, e_i)}$ $= 0 \quad \forall v \in V$.

$$= f(\underbrace{\sum_i c_i e_i}_{u}) \Rightarrow u = 0 \text{ since } f \text{ is nondeg.}$$

But $\{e_i\}$ is a basis $\Rightarrow \forall i, \ c_i = 0$.

$\Rightarrow \{e_i^*\}$ are lin. indep.

② (If $\{e_i^*\}$ is linearly indep)

Similar.

<u>defn</u> A symmetric bilinear form on V over $\mathbb{R}$

- <u>positive definite</u> if $f(v,v) > 0 \quad \forall v \neq 0$
- <u>negative definite</u> if $f(v,v) < 0 \quad \forall v \neq 0$.

<u>Rmk</u> Symmetric matrices have real eigenvalues, and are diagonalizable.

$\Rightarrow$ Given any symmetric bilinear form on $\mathbb{R}^d$, we can choose a basis so that the matrix is

$$\left.\begin{array}{c|c|c} I_p & & \\ \hline & -I_n & \\ \hline & & O_r \end{array}\right\} \text{total dimn } d.$$

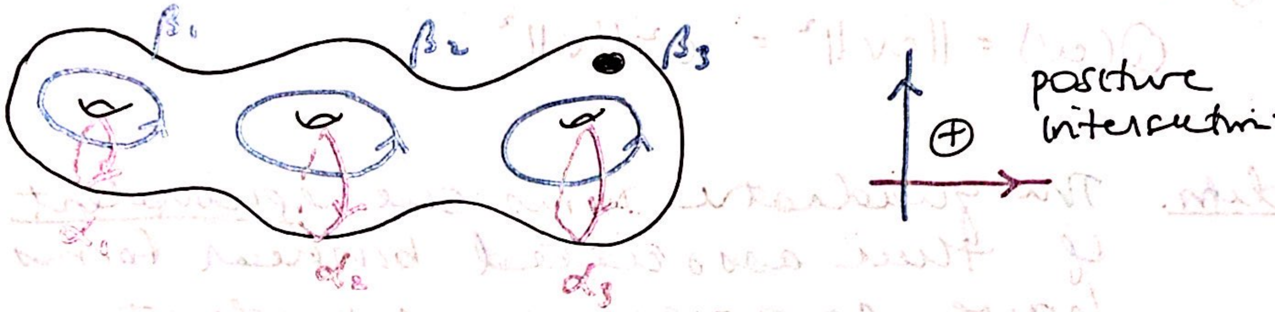
i.e. the vector space V $(\cong \mathbb{R}^d)$ decomposes as $V = W_+ \oplus W_- \oplus W_0$ where $f|_{W_+}$ is pos definite, $f|_{W_-}$ is neg. def,

<u>defn.</u> The <u>signature</u> of a symmetric bilinear form f is $\sigma = p - n$.
(# pos eigenvalues − # neg. eigenvalues).

<u>Fact</u> 2 symm. real $n \times n$ matrices are congruent iff they have the same <u>rank</u> and signature.
$p+n$.

→ <u>Quadratic Forms section here</u>.

eg. Seifert form.



$f(\gamma_1, \gamma_2)$ = algebraic intersection # of $\gamma_1 \cdot \gamma_2$.

Quadratic Forms

defn. Let V be a vector space over a field k. A quadratic form is a function $Q: V \longrightarrow k$ such that

- $Q(cv) = c^2 Q(v) \quad \forall v \in V, c \in k$.
- $f: V \times V \longrightarrow k$ defined by
  $f(u,v) = Q(u+v) - Q(u) - Q(v)$
  is a bilinear form.

↳ note The associated bilinear form $f(u,v)$ is necessarily symmetric.

It turns out nondegenerate ~~symmetric~~ alternating forms also have a nice form if you can find the right basis:

<u>Fact.</u> If f is nondegenerate and alternating, then

$$V \cong H_1 \oplus H_2 \oplus \cdots \oplus H_m$$

where each H_i has inner product matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

<u>Rmk</u> Each H_i is called a <u>hyperbolic plane</u>. The matrix

$$\mathbb{H} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$ is often called the <u>hyperbolic form</u>.

<u>defn.</u> A <u>symplectic basis</u> for a ~~an alternating~~ nondegenerate (V, f) is a basis $\{x_1, y_1, \ldots, x_m, y_m\}$ such that

$$f(x_i, y_i) = 1, \quad f(y_i, x_i) = -1 \quad \forall i$$

& all other pairings are 0:

$$0 = f(x_i, x_i), f(y_i, y_i), f(x_i, y_j), f(y_i, x_j)$$