

Lecture 18

- ① finish symm bilinear forms, quadratic forms
 - ② some alt. forms
 - ③ Tensor + Exterior algebras.
- Bonus? Super fast intro to differential forms.

let k be a commutative ring.

defn. A k -algebra A is a graded k -algebra if there are k -submodules A^p for $p \geq 0$ such that

$$\textcircled{1} A = \bigoplus_{p \geq 0} A^p$$

$$\textcircled{2} \text{ for all } p, q \geq 0, \quad A^p A^q \subseteq A^{p+q}$$

An element $x \in A^p$ is called a homogenous element of degree p .

eg. $k[X] = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} k \cdot X^i$, $k[X]/(X^2) = k \cdot 1 + k \cdot X$, $k[X_1, X_2, \dots, X_n]$

eg. Cohomology ring $H^*(X, k)$, under cup product

defn. If A and B are graded k -algebras, and $d \in \mathbb{Z}$,

a graded map of degree d is a k -algebra map

$$f: A \rightarrow B \text{ such that } f(A^p) \subseteq B^{p+d} \text{ for all } p \geq 0.$$

note a "graded map" or "grading-preserving map" means a degree 0 map.

The main examples we want to discuss:

- Tensor algebras. $T(M)$
- Exterior algebras $\Lambda(M)$

Tensor algebras of modules

defn let $M \in k\text{-mod}$ eg. $V \in \text{Vect}_{\mathbb{F}}$

The tensor algebra of M is the graded algebra over k

where

$$\left. \begin{array}{l} T^0(M) = k \\ T^1(M) = M \\ T^2(M) = M \otimes M \\ \text{etc.} \end{array} \right\} \text{u. } T(M) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \left(\bigotimes^n M \right)$$

Multiplication is induced by

$$\underbrace{(a_1 \otimes a_2 \otimes \dots \otimes a_k)}_{\bigotimes^k M}, \underbrace{(b_1 \otimes b_2 \otimes \dots \otimes b_\ell)}_{\bigotimes^\ell M} \mapsto \underbrace{\left(\bigotimes_{i=1}^k a_i \right) \otimes \left(\bigotimes_{j=1}^\ell b_j \right)}_{\bigotimes^{k+\ell} M}$$

eg. in the wild: Bar construction / Standard complex

\rightsquigarrow projective (\Rightarrow flat) resolution.

\rightsquigarrow allow you to compute Hochschild homology, etc.

Exterior Algebras of modules

defn. let $M \in k\text{-mod}$. Its exterior algebra is

$$\Lambda(M) = T(M)/J$$

where J is the 2-sided ideal generated by all $m \otimes m$ for $m \in M$:

$$e. J = \{ a \otimes m \otimes b : a, b \in T(M), m \in M \}$$

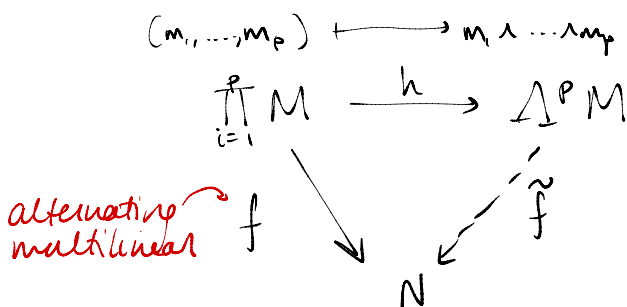
The coset $(m_1 \otimes m_2 \otimes \dots \otimes m_p) + J$ in $\Lambda(M)$ is denoted $m_1 \wedge m_2 \wedge \dots \wedge m_p$ (wedge of p factors)

Fact. $\Lambda(M) = k \oplus M \oplus \Lambda^2 M \oplus \Lambda^3 M \oplus \dots$

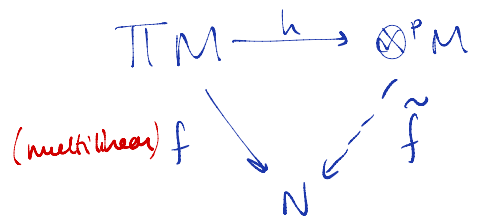
defn. $\Lambda^p M$ is called the p^{th} exterior power of M .

defn. A k -multilinear fn. $f: \prod_{i=1}^p M \rightarrow N$ is alternating if $f(m_1, \dots, m_p) = 0$ whenever $m_i = m_j$ for some $i \neq j$.

Universal Property



Compare:



Theorem (Anticommutativity) let $x \in \Delta^p M$, $y \in \Delta^q M$.

Then $x \wedge y = (-1)^{pq} y \wedge x$.

pf. Use double induction.

$p=0$ or $q=0$: either x or $y \in \mathcal{h} \dots$ Then $x \wedge y = y \wedge x$.

Base case: $p=1, q=1$

$$0 = (x+y) \wedge (x+y) = \cancel{x \wedge x} + x \wedge y + y \wedge x + \cancel{y \wedge y} \quad \checkmark$$

(Continue w/ double induction). //