Lecture 23

<u>defn</u> The field $\overline{F}$ is called an <u>algebraic closure</u> of F if $\overline{F}$ is algebraic over F, and every $f(x) \in F[x]$ splits over $\overline{F}$.

<u>defn</u> A field K is <u>algebraically closed</u> if every polynomial with coeff in K has a root in K.

↳ note that this means $f(x) \in K[x]$ must have <u>all</u> roots in K (there are no degree ≥ 2 irred polyns over K!)

↳ $\Rightarrow \overline{F} = F$

<u>prop</u> The alg closure $\overline{F}$ of F is algebraically closed.

pf. Let $f(x) \in \overline{F}[x]$, and let α be a root of $f(x)$ in an extension of $\overline{F}$. Then α is algebraic over F:

details not covered in class

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ where $a_i \in \overline{F}$ is already a polynomial in $F(a_0, a_1, \dots, a_n)[x]$ which is a finite and therefore algebraic extension of F.

Let $K = F(a_0, \dots, a_n)$. Then $K(\alpha)/K$ is also finite (degree $\leq n!$)

$\Rightarrow K(\alpha)/F$ is finite and thus algebraic.

$\Rightarrow \alpha \in \overline{F}$ by defn. of $\overline{F}$.

$\Rightarrow$ Any $f(x) \in \overline{F}[x]$ contains a root in $\overline{F}$ $\Rightarrow \overline{F}$ is alg'ly closed. ☒

Prop. For any field F, there exists an algebraically closed field K containing F.

Note: let $F = \mathbb{Q}$. K could be $\overline{\mathbb{Q}}$ or $\mathbb{C}$. $\overline{\mathbb{Q}} \subsetneq \mathbb{C}$ since π is not alg. over $\mathbb{Q}$.

pf.

- For every monic, irred $f(x) \in F[x] \rightsquigarrow$ define variable x_f.

  Form a huge polyn ring $F[\{x_f\}]$

- Define the ideal I generated by all the $\{f(x_f)\}$ in $F[\{x_f\}]$.

  Claim: I is proper.

  BWOC, suppose $1 \in I$. Then there is a relation

  What does it look like? $c_1 f_1(x_{f_1}) + c_2 f_2(x_{f_2}) + \dots + c_n f_n(x_{f_n}) = 0$.

  What are the c_i? $c_i \in F[\{x_f\}] \Rightarrow$ finite polynomials.

  The relation is finite, so we can relabel the variables so that it looks like

  $$g_1(x_1, x_2, \dots, x_m) f_1(x_1) + \dots + g_n(x_1, x_2, \dots, x_m) f(x_n) = 1$$

  where

  - For $1 \le i \le n$, $x_i = x_{f_i}$, and the rest are the finitely many other x_{f_j} that appear in the "c_i" (now g_i) polynomials.

  Let F' be a finite extension of F containing a root α_i of each $f_i(x)$ $(i = 1, 2, \dots, n)$.

  Evaluating the relation above at $\{x_i = \alpha_i\}$ over F', we have $0 = 1$ in F'. ↯.

  So I is proper, and therefore contained in a maximal ideal M.

  (Zorn's lemma)

- Then $K_1 = F[\{x_f\}]/M$ is a field containing an isom copy of F.
  - Each f has a root in K_1, by construction (namely $\overline{x_f}$).
  - Now rinse and repeat to get K_2, and so on.
  - eventually, since every polynomial is finite degree, every polynomial will split over some finite K_d.
- Let $K = \bigcup_{j \geq 0} K_j$ $\quad (K_0 = F)$ $\quad$ Check: K contains (an isom copy of) F ✓

  and every polyn in $F[x]$ splits. ✓

$\boxtimes$

<u>Rmk</u> If you already have a "universe" that F lives in (eg. $\mathbb{Q} \subset \mathbb{C}$), ie F is already contained in an algebraically closed field K, then the "best" one (smallest alg'ly closed field containing F) is $\overline{F}$, the collection of elements in K that are algebraic over F.

<u>ex</u> Check that $\overline{F}$ is alg'ly closed.

<u>Fact</u> $\overline{F}$ is unique up to isom.

<u>thm.</u> (Fundamental Theorem of Algebra) $\mathbb{C}$ is alg'ly closed.

Separable + Inseparable Extensions

For HW09: just need the word separable

Weird stuff happens over number fields of positive characteristic:

eg. $x^2 - t$ over the field $F = \mathbb{F}_2(t)$ rational fns in t w/ coeffs in $\mathbb{F}_2$

- $f(x) = x^2 - t$ is irred. Why?

  $\mathbb{F}_2[t]$ is a PID. $f(x)$ is Eisenstein at (t).

  $\Rightarrow$ irred over $\mathbb{F}_2[t]$, and also over $\mathrm{Frac}(\mathbb{F}_2[t]) = \mathbb{F}_2(t)$.

- Let $\sqrt{t}$ be a root of $f(x)$ in an extension field.

  Then $(x - \sqrt{t})^2 = x^2 - 2x\sqrt{t} + t = x^2 + t = x^2 - t$.

  $\Rightarrow \sqrt{t}$ is a root of multiplicity 2 of the irred polyn. $f(x)$!

defn. Let $f(x) \in F[x]$. Let K be a splitting field of $f(x)$.

- If all roots of $f(x)$ are distinct (ie all multiplicity 1),

  then $f(x)$ is <u>separable</u>.

- Otherwise, $f(x)$ is <u>inseparable</u>.

So $x^2 - t \in \mathbb{F}_2(t)[x]$ is inseparable.