Mat 250B HW02

①

(a) $0 \longrightarrow \ker f \xhookrightarrow{i} M \xrightarrow{f} N \xrightarrow{p} \operatorname{coker} f \longrightarrow 0$

inclusion map

$n \longmapsto [n + \operatorname{im} f]$ (surjective)

Check exactness.

(b) $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$

$\operatorname{im} f = B \Leftrightarrow \ker g = B.$

$\Leftrightarrow \operatorname{im} g = 0 \Leftrightarrow \ker h = 0 \Leftrightarrow h$ is inj.

② Let $M \in {}_R\mathrm{Mod}$, $M \neq 0$.

<u>cyclic + gen'd by any nonzero element $\Rightarrow$ irred.</u>

suppose $0 \neq N \subset M$ submodule.

let $n \in N$, $n \neq 0$. Then since $n \in M$, n generates M (by assumption) so $\{rn \mid r \in R\} = M \Rightarrow N = M$.

<u>irred $\Rightarrow$ cyclic</u>: We'll show the contrapositive.

Suppose M is not cyclic, or cyclic but not able to be generated by all nonzero elements. In any case $\exists$ some $m \in M$, $m \neq 0$, such that $0 \neq \{rm \mid r \in R\} =: N \subsetneq M$.

Then N is a (nonzero) proper submodule of M.

$\Rightarrow M$ is not irred.

$\{\mathbb{Z}/p\mathbb{Z}$ where p prime$\}$ = all irred $\mathbb{Z}$-mods.

③ (a) Let $n = |G|$, and label $G = \{g_1, \dots, g_n\}$.

Then $kG = k[G] = \{ \sum_{i=1}^{n} c_i g_i \mid c_i \in k \}$.

Recall $(kG)^{op}$ has multiplication in the opp. order.

(k is a field so coefficients commute with all g_i)

Let $a, b \in k$, and $g, h \in G$.

In $k[G]$, $\mu(ag, bh) = agbh = ab(gh)$.

In $k[G]^{op}$, $\mu_{op}(ag, bh) = bhag = ab(hg)$

Define $\varphi : k[G] \longrightarrow k[G]^{op}$ by extending $g \mapsto g^{-1}$ k-linearly.

$\sum c_i g_i \mapsto \sum c_i g_i^{-1}$

- φ is a group hom by construction.
- Since $1_{k[G]} = 1_k \cdot 1_G$, and $1_G^{-1} = 1_G$, $\varphi(1_{kG}) = 1_{kG}$.
- Then $\varphi(ag \cdot bh) = \varphi(ab(gh)) = ab\,\varphi(gh) = ab h^{-1} g^{-1}$.

  and $\varphi(ag) * \varphi(bh) = \varphi(bh)\varphi(ag) = bh^{-1} a g^{-1} = ab h^{-1} g^{-1}$.
- φ^{-1} exists (also defined by $g \mapsto g^{-1}$); φ is an isom.

(b) $\mathbb{H} = \mathbb{R}[Q]$ where $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, the quaternion group.

(5) Let $A, B \in \mathrm{Grp}$.

(a)

$$\begin{tikzcd} & X \arrow[dl, bend right, "f"'] \arrow[d, "\theta"] \arrow[dr, bend left, "g"] & \\ A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

Let $\theta(x) = (f(x), g(x))$.

Check θ is unique.

(b)

$$\begin{tikzcd} A \arrow[dr, "i_1"] \arrow[ddr, bend right, "f"'] & & B \arrow[dl, "i_2"'] \arrow[ddl, bend left, "g"] \\ & A * B \arrow[d, "\theta"] & \\ & X & \end{tikzcd}$$

$A * B$ is generated by the elements of $A \cup B$.

So we can define θ by

$\theta(a) = f(a) \quad \forall a \in A$

and $\quad \theta(b) = g(b) \quad \forall b \in B$.

and extend to a group hom.

Then θ is defined on all words $w \in A * B$

Check that θ is unique.

(6)

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[l, bend right=50, "f^{-1}"'] \arrow[l, bend left=50, "g"] \end{tikzcd}$$

Suppose $f^{-1}f = id_A = gf$

and $\quad ff^{-1} = id_B = fg$.

If $gf = id_A = f^{-1}f$, then $\quad gff^{-1} = f^{-1}ff^{-1}$,

$\Rightarrow g \cdot id_B = id_A f^{-1} \Rightarrow g = f^{-1}$.

⑦ (a) This follows by induction and the fact that

$$A \oplus B \cong A \times B \quad \text{for} \quad A, B \in {}_R\mathrm{Mod}.$$

Note that they really are the same module; it's just that as a coprod (resp. product), $A \oplus B$ (resp $A \times B$) comes with injection (resp. proj'n) maps.

(b) See notes; formalize the argument.