

Mat 250B HW02

①

$$(a) \quad 0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{P} \operatorname{coker} f \longrightarrow 0$$

$\uparrow$   
 inclusion map

$\uparrow$   
 $n \mapsto [n + \operatorname{im} f]$   
 (surjective)

Check exactness.

$$(b) \quad A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

$$\operatorname{im} f = B \Leftrightarrow \ker g = B.$$

$$\Leftrightarrow \operatorname{im} g = 0 \Leftrightarrow \ker h = 0 \Leftrightarrow h \text{ is inj.}$$

② Let  $M \in R\text{-Mod}$ ,  $M \neq 0$ .

cyclic + gen'd by any nonzero element  $\Rightarrow$  irred.

suppose  $0 \neq N \subset M$  submodule.

let  $n \in N$ ,  $n \neq 0$ . Then since  $n \in M$ ,  $n$  generates  $M$

(by assumption) so  $\{rn \mid r \in R\} = M \Rightarrow N = M$ .

irred  $\Rightarrow$  cyclic: We'll show the contrapositive.

Suppose  $M$  is not cyclic, or cyclic but not able to be generated by all nonzero elements. In any case  $\exists$  some

$m \in M$ ,  $m \neq 0$ , such that  $0 \neq \{rm \mid r \in R\} =: N \subsetneq M$ .

Then  $N$  is a (nonzero) proper submodule of  $M$ .

$\Rightarrow M$  is not irred.

$\{\mathbb{Z}/p\mathbb{Z} \text{ where } p \text{ prime}\} = \text{all irred } \mathbb{Z}\text{-mods.}$

③ (a) Let  $n = |G|$ , and label  $G = \{g_1, \dots, g_n\}$ .

$$\text{Then } kG = k[G] = \left\{ \sum_{i=1}^n c_i g_i \mid c_i \in k \right\}.$$

Recall  $(kG)^{\text{op}}$  has multiplication in the opp. order.

( $k$  is a field so coefficients commute with all  $g_i$ )

Let  $a, b \in k$ , and  $g, h \in G$ .

$$\text{In } k[G], \quad \mu(ag, bh) = agbh = ab(gh).$$

$$\text{In } k[G]^{\text{op}}, \quad \mu_{\text{op}}(ag, bh) = bhag = ab(hg)$$

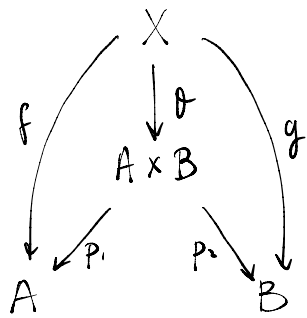
Define  $\varphi: k[G] \rightarrow k[G]^{\text{op}}$  by extending  $g \mapsto g^{-1}$   
 $\sum c_i g_i \mapsto \sum c_i g_i^{-1}$   $k$ -linearly.

- $\varphi$  is a group hom by construction.
- Since  $1_{k[G]} = 1_k \cdot 1_G$ , and  $1_G^{-1} = 1_G$ ,  $\varphi(1_{kG}) = 1_{kG}$ .
- Then  $\varphi(ag \cdot bh) = \varphi(ab(gh)) = ab\varphi(gh) = abh^{-1}g^{-1}$ .  
and  $\varphi(ag) * \varphi(bh) = \varphi(bh)\varphi(ag) = bh^{-1}ag^{-1} = abh^{-1}g^{-1}$ .
- $\varphi^{-1}$  exists (also defined by  $g \mapsto g^{-1}$ );  $\varphi$  is an isom.

(b)  $H = \mathbb{R}[Q]$  where  $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ , the quaternion group.

5) Let  $A, B \in \text{Grp}$ .

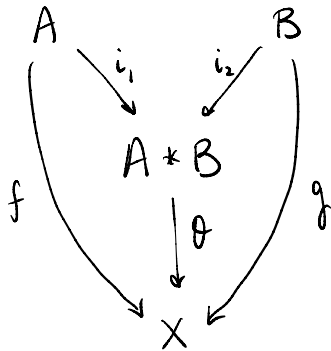
(a)



Let  $\theta(x) = (f(x), g(x))$ .

Check  $\theta$  is unique.

(b)



$A * B$  is generated by the elements of  $A \cup B$ .

So we can define  $\theta$  by

$$\theta(a) = f(a) \quad \forall a \in A$$

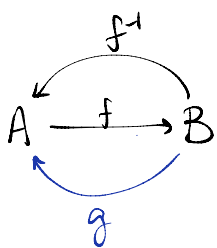
$$\text{and } \theta(b) = g(b) \quad \forall b \in B.$$

and extend to a group hom.

Then  $\theta$  is defined on all words  $w \in A * B$

Check that  $\theta$  is unique.

6)



Suppose  $f^{-1}f = \text{id}_A = gf$

and  $ff^{-1} = \text{id}_B = fg$ .

If  $gf = \text{id}_A = f^{-1}f$ , then  $gf f^{-1} = f^{-1}f f^{-1}$ ,

$$\Rightarrow g \cdot \text{id}_B = \text{id}_A f^{-1} \Rightarrow g = f^{-1}.$$

⑦ (a) This follows by induction and the fact that

$$A \oplus B \cong A \times B \quad \text{for} \quad A, B \in {}_R \text{Mod.}$$

Note that they really are the same module; it's

just that as a coprod (resp. product),

$A \oplus B$  (resp.  $A \times B$ ) comes with injection (resp. proj'n)

maps.

(b) See notes; formalize the argument.