

HW03

$$\textcircled{1} \quad 0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0 \quad \text{exact}$$

$$\text{Hom}_R(C, M) \xrightarrow{\beta^*} \text{Hom}_R(B, M)$$

$$[f: C \rightarrow M] \mapsto \beta^*(f) = [f\beta: B \xrightarrow{\beta} C \xrightarrow{f} M]$$

WTS β^* is injective:

Suppose $\beta^*(f) = \beta^*(g)$, i.e. $f\beta = g\beta$. Let $c \in C$.

Choose any $b \in \beta^{-1}(c)$ (exists as β surjective).

Then $f\beta(b) = g\beta(b) \Rightarrow f(c) = g(c)$. \leftarrow

$\textcircled{2}$ \Rightarrow Assume $P_1 \oplus P_2$ projective. Then \exists free module F

and module K where $F \cong K \oplus (P_1 \oplus P_2)$,

so both P_1, P_2 are also summands of F .

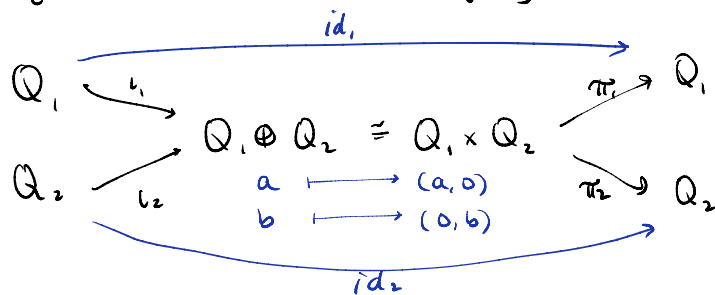
\Leftarrow Assume P_1, P_2 projective, where

$$F_1 \cong K_1 \oplus P_1, \quad F_2 \cong K_2 \oplus P_2 \quad (F_1, F_2 \text{ free})$$

Then $P_1 \oplus P_2$ is a summand of the free module

$F_1 \oplus F_2$. (generated by $B = B_1 \cup B_2$, where B_i is a basis for F_i).

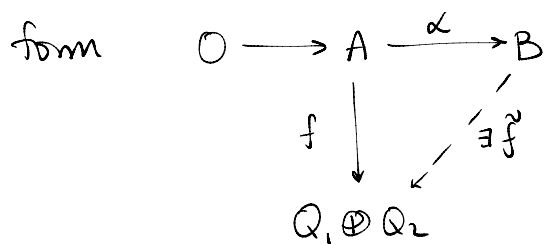
- ③ Recall that for modules, $Q_1 \oplus Q_2 \cong Q_1 \times Q_2$
 and we can define projection to each coordinate
 (even just on the underlying sets).



Below, let $0 \rightarrow A \xrightarrow{\alpha} B$ be an exact sequence.

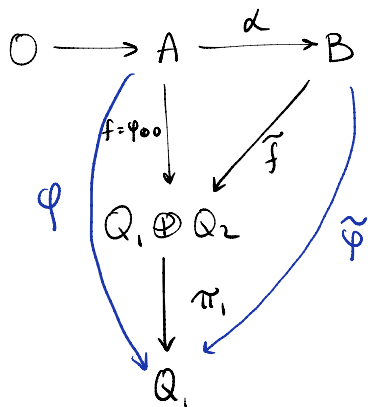
\Rightarrow Assume $Q_1 \oplus Q_2$ is injective.

Given $\psi: A \rightarrow Q_1$,



where $f = \psi \oplus 0$,
 i.e. $f(a) = \psi(a) + 0$.

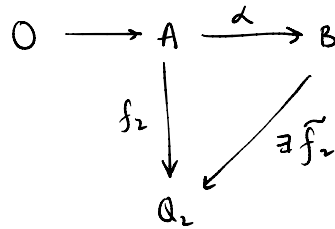
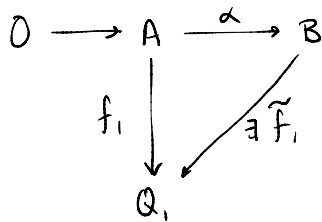
Then $\exists \tilde{f}$ lifting f . Then $\tilde{\psi} = \pi_1 \tilde{f}$ lifts ψ :



⊖ Assume Q_1, Q_2 are exact.

Given $\varphi: A \rightarrow Q_1 \oplus Q_2$, consider the maps $(j=1, 2)$

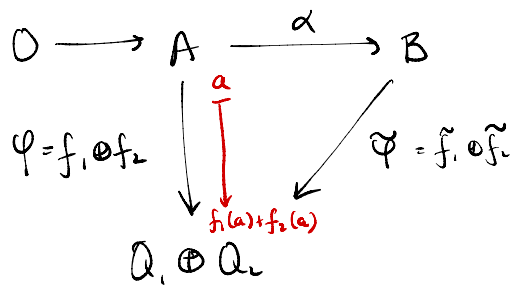
$f_j: A \rightarrow Q_j$ where $f_j = \pi_j \varphi$.



} Q_i are injective

let $\tilde{\varphi} = \tilde{f}_1 \oplus \tilde{f}_2: B \rightarrow Q_1 \oplus Q_2$
 $b \mapsto f_1(b) + f_2(b)$

Then check that $\tilde{\varphi}$ lifts φ :



④ Suppose (i) every module is projective.

Then given any SES

$$0 \rightarrow Q \rightarrow M \rightarrow N \rightarrow 0,$$

N is projective \Rightarrow SES splits.

$\Rightarrow Q$ is injective.

Suppose (ii) every module is injective.

Then given any SES

$$0 \rightarrow L \rightarrow M \rightarrow P \rightarrow 0,$$

L is injective \Rightarrow SES splits

$\Rightarrow P$ is projective.

⑤ By classification of finitely generated \mathbb{Z} -mod,
 $A \cong \mathbb{Z}/r_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/r_n\mathbb{Z}$. (may assume all $r_i > 1$)

(a) By Ex. 2, $\mathbb{Z}/r\mathbb{Z}$ is not projective.

$0 \rightarrow r\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/r\mathbb{Z} \rightarrow 0$ does not split.

because $r\mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z} \neq \mathbb{Z}$.

(b) By Ex 3, $\mathbb{Z}/r\mathbb{Z}$ is not injective.

Indeed, it is not divisible, since

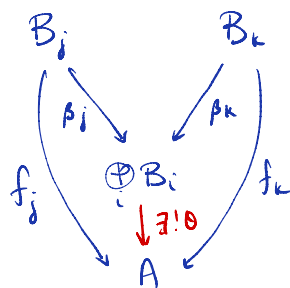
$$r \cdot (\mathbb{Z}/r\mathbb{Z}) = \{0\}.$$

②

(a) We'll construct an isomorphism:

$$\prod_{i \in I} \text{Hom}_R(B_i, A) \xrightarrow{\Phi} \text{Hom}_R\left(\bigoplus_{i \in I} B_i, A\right)$$

$f = (f_i)_{i \in I} \quad \longmapsto \quad \mathcal{O}_f$
given by univ. property.



(\mathcal{O} is univ. \Rightarrow this is well-defined)

(By inspection, this respects "+" because if we are given $f = (f_i)$ and $g = (g_i)$, $\mathcal{O}_f + \mathcal{O}_g$ is a map that makes the corresponding diagram commute.)

Now define a candidate for the inverse map:

$$\text{Hom}_R\left(\bigoplus_{i \in I} B_i, A\right) \xrightarrow{\Psi} \prod_{i \in I} \text{Hom}_R(B_i, A)$$

$[F: \bigoplus_{i \in I} B_i \rightarrow A] \longmapsto (F\beta_i)_{i \in I}$

$(B_i \xrightarrow{\beta_i} \bigoplus_{i \in I} B_i \xrightarrow{F} A)$

Finally, check that ① $\Psi\Phi = \text{id}$ on $\prod \text{Hom}_R(B_i, A)$

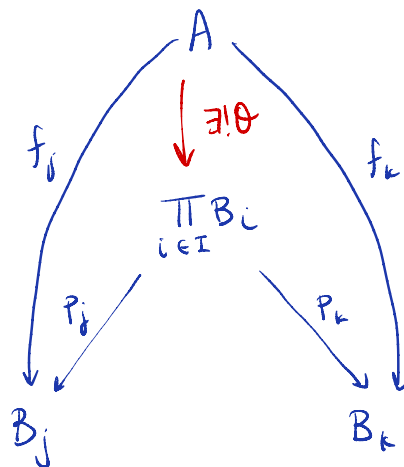
and ② $\Phi\Psi = \text{id}$ on $\text{Hom}_R(\bigoplus B_i, A)$

(b) Construct an isom:

$$\prod_{i \in I} \text{Hom}_R(A, B_i) \xrightarrow{\Phi} \text{Hom}_R(A, \prod_{i \in I} B_i)$$

$$f = (f_i)_{i \in I} \longmapsto \theta_f \text{ defined by U.P.}$$

again, this is well-defined
because θ_f is unique



Candidate Inverse Ψ :

$$\text{Hom}_R(A, \prod_{i \in I} B_i) \xrightarrow{\Psi} \prod_{i \in I} \text{Hom}_R(A, B_i)$$

$$\left[F: A \rightarrow \prod_{i \in I} B_i \right] \longmapsto (p_i F)_{i \in I}$$

$$\left(A \xrightarrow{F} \prod_{i \in I} B_i \xrightarrow{p_i} B_i \right)$$

Again, check (by inspection) that these are
abelian group homomorphisms
and that Ψ and Φ really are inverses.

⑦ (Exercise effectively removed from HW)

$$\text{Let } F = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}$$

$$F^* = \text{Hom}_{\mathbb{Z}}\left(\bigoplus_{n \in \mathbb{N}} \mathbb{Z}, \mathbb{Z}\right) \cong \prod_{n \in \mathbb{N}} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \prod_{n \in \mathbb{N}} \mathbb{Z}.$$

ISTS $\prod_{n \in \mathbb{N}} \mathbb{Z}$ is not free.

By way of contradiction, suppose $\prod_{n \in \mathbb{N}} \mathbb{Z}$ were free.

(Solution uses cardinality of $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}$ and $\prod_{n \in \mathbb{N}} \mathbb{Z}$
and is quite involved)

⑧ The question asks when $\text{Hom}_R(-, R)$ is an exact functor. This happens precisely when R is an injective R -module!

(There are ways to characterize this, but none that I know of that only reference the ring structure of R .)