HW04

Pf idea [DF 10.5.15 - 17]

(a) Ring R, module $M \in {}_R\mathrm{Mod}$ are both in particular $\mathbb{Z}$-modules.

<u>Claim</u> $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is an R-module.

Let $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(R,M)$, ie $\varphi : R \to M$ preserves addition

Then let $r \cdot \varphi : R \to M$ be the map $x \mapsto \varphi(xr)$

<u>Check</u> R-action

① distributive:

$$r \cdot (\varphi + \psi)(x) = (\varphi + \psi)(xr) = \varphi(xr) + \psi(xr) = (r \cdot \varphi + r \cdot \psi)(x).$$

$$\begin{aligned}(r_1 + r_2) \cdot \varphi(x) &= \varphi(x(r_1 + r_2)) \\ &= \varphi(xr_1 + xr_2) \\ &= \varphi(xr_1) + \varphi(xr_2) \\ &= r_1 \cdot \varphi(x) + r_2 \cdot \varphi(x)\end{aligned}$$

② associative:

$$(r_1 r_2)\varphi(x) = \varphi(x \cdot r_1 r_2) = \varphi((xr_1)r_2) = r_2 \cdot \varphi(xr_1) = r_1 \cdot r_2 \varphi(x).$$

③ identity: $1_R \cdot \varphi(x) = \varphi(x \cdot 1_R) = \varphi(x).$

(b)

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists \tilde{f}"] \\ & M & \end{tikzcd} \implies \begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "F"'] & B \arrow[dl, dashed, "\exists \tilde{F}"] \\ & \mathrm{Hom}_{\mathbb{Z}}(R,M) & \end{tikzcd}$$

Let $F: A \to \mathrm{Hom}_{\mathbb{Z}}(R,M)$. Want $f: A \to M$..

$a \mapsto [F(a): R \to M]$

Define $f: A \to M$

$a \mapsto F(a)(1_R)$

Check this is a $\mathbb{Z}$-mod hom:

$f(a_1+a_2) = F(a_1+a_2)(1_R) = F(a_1)(1_R) + F(a_2)(1_R) = f(a_1)+f(a_2)$

F is an R-module hom.

$\Rightarrow \exists \tilde{f}: B \to M.$

Define $\tilde{F}: B \to \mathrm{Hom}_{\mathbb{Z}}(R,M)$ by

$b \mapsto [\tilde{F}(b): R \to M]$

$r \mapsto \tilde{f}(rb)$

ie $\tilde{F}(b)(r) = \tilde{f}(rb)$

① $\tilde{F}(b)$ is a $\mathbb{Z}$-mod hom (b/c distributivity of R-action on B, etc)

② $\tilde{F}$ is R-mod hom:

$\tilde{F}(b_1+b_2)(r) = \tilde{f}(r(b_1+b_2)) \overset{B \text{ is } R\text{-mod}}{=} \tilde{f}(rb_1+rb_2) \overset{\tilde{f} \text{ is } R\text{-mod hom}}{=} \tilde{f}(rb_1)+\tilde{f}(rb_2)$

$= \tilde{F}(b_1)(r) + \tilde{F}(b_2)(r)$

$\tilde{F}$ actually is a lift of F: WTS $\tilde{F}\alpha = F$:

$$\begin{tikzcd} & a \arrow[r, mapsto] & \alpha(a) \\ 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "F"'] & B \arrow[dl, dashed, "\exists \tilde{F}"] \\ F(a) & \mathrm{Hom}_{\mathbb{Z}}(R,M) & \tilde{F}(\alpha(a)) \end{tikzcd}$$

$\tilde{F}(\alpha(a))(r) = \tilde{f}(r\alpha(a)) \overset{\tilde{f} \text{ is } R\text{-map}}{=} r\tilde{f}(\alpha(a))$

$= rf(a) = r \cdot F(a)(1_R) = F(a)(r \cdot 1_R)$

$= F(a)(r)$ ✓

(c) If Q is an injective R-mod, then $\mathrm{Hom}_{\mathbb{Z}}(R,M)$ is also.
(follows from (b)).

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② (a) Corollary from class; view M as $\mathbb{Z}$-module.

(b)

<u>Claim</u> $\mathrm{Hom}_R(R,M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R,M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R,Q)$.

Pf. (These are all Ab groups).

Being R-linear is more restrictive than being $\mathbb{Z}$-linear.

$\mathrm{Hom}_{\mathbb{Z}}(R,-)$ is left exact. //

(c)

<u>Claim</u> $M \cong \mathrm{Hom}_R(R,M)$ as R-mods

Pf.
$$\begin{aligned} \Phi: M &\longrightarrow \mathrm{Hom}_R(R,M) \\ m &\longmapsto f_m: R \longrightarrow M \\ &\qquad\quad 1_R \longmapsto m. \end{aligned}$$
(check R-hom, isom)

$\Rightarrow M \cong \mathrm{Hom}_R(R,M) \subset \mathrm{Hom}_{\mathbb{Z}}(R,Q)$ which is injective
as a $\mathbb{Z}$-module.

WTS $\mathrm{Hom}_{\mathbb{Z}}(R,Q)$ also injective as an R-module

See Canvas Announcement.

③

(a) Let $a \in G$, $a \neq 0$.

If $|a| = n < \infty$, define $f(a) = 1/n + \mathbb{Z}$

If $|a| = \infty$, define $f(a) = 1/2 + \mathbb{Z}$.

(or any nonzero element in $\mathbb{Q}/\mathbb{Z}$).

This induces a map $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}a, \mathbb{Q}/\mathbb{Z})$

($\mathbb{Z}a$ is the cyclic submodule of G generated by a, otherwise known as $\langle a \rangle \leq G$)

We now have

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}a = \langle a \rangle \arrow[r, hook] \arrow[d, "f"] & G \arrow[dl, "\tilde{f}"] \\ & \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

where $\mathbb{Q}/\mathbb{Z}$ is injective (since it is a quotient of an injective module over a PID)

So $\tilde{f}$ exists.

In particular, $\tilde{f}(a) = f(a) \neq 0$.

(b) Since $\mathbb{Q}/\mathbb{Z}$ is injective (as quotient module of the injective $\mathbb{Z}$-mod $\mathbb{Q}$), $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Q}/\mathbb{Z})$ is exact.

Note this is a contravariant Hom functor.

(c) $G = \mathbb{Z}/r_1\mathbb{Z} \times \mathbb{Z}/r_2\mathbb{Z} \times \cdots \times \mathbb{Z}/r_n\mathbb{Z}$.

$$G^* = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}\left(\prod_{i=1}^{n} \mathbb{Z}/r_i\mathbb{Z}, \mathbb{Q}/\mathbb{Z}\right)$$

$$\cong \bigoplus_{i=1}^{n} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/r_i\mathbb{Z}, \mathbb{Q}/\mathbb{Z}).$$

Recall Each Hom gp is an Ab group; finite $\oplus \cong$ finite Π.

So it suffices to show that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/r\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/r\mathbb{Z}$$

Since each $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/r\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is determined by $f(1)$, the isomorphism is given by $f \longmapsto f(1)$.