# MAT 250B HW04

[add your name here]

Due Friday, 2/2/24 at 11:59 pm on Gradescope

**Reminder** Your homework submission **must be typed** (TeX'ed) up in full sentences, with proper mathematical formatting.

**Notes**

- We will continue to write "$R$-module" as shorthand for "left $R$-module".
- We talked about pullbacks in class, but we didn't get to pushouts. You can read about this dual construction in the book or in the posted lecture notes.

## Exercise 1

Let $R$ be a ring and let $M \in {}_R\mathbf{Mod}$.

(a) Prove that $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is a left $R$-module under the action $(r\varphi)(x) = \varphi(xr)$ where $r, x \in R$.

(b) Let $0 \to A \xrightarrow{\alpha} B$ is an exact sequence of $R$-modules. Prove that if every $R$-map $f : A \to M$ lifts to an $R$-map $\tilde{f} : B \to M$ where $f = \tilde{f} \circ \alpha$, then every $R$-map $F : A \to \mathrm{Hom}_{\mathbb{Z}}(R, M)$ lifts to an $R$-map $\tilde{F} : B \to \mathrm{Hom}_{\mathbb{Z}}(R, M)$.

*Hint:* Given $F$, show that $f(a) = F(a)(1_R)$ defines an $R$-map from $A$ to $M$. Then show that $\tilde{F}(b)(r) = \tilde{f}(rb)$ defines an $R$-map from $B$ to $\mathrm{Hom}_{\mathbb{Z}}(R, M)$. Finally, check that $\tilde{F}$ lifts $F$.

(c) Prove that if $Q$ is an injective $R$-module, then $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ is also an injective $R$-module.

## Exercise 2

Let $R$ be a ring. In this exercise you will prove the following theorem.

**Theorem.** Every $R$-module $M$ is contained in an injective $R$-module.

(a) Show that $M$ (viewed as a $\mathbb{Z}$-module) is contained in an injective $\mathbb{Z}$-module $Q$.

(b) Prove that $\mathrm{Hom}_R(R, M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R, M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R, Q)$.

(c) Prove that $M \cong \mathrm{Hom}_R(R, M)$ as $R$-modules, and then conclude that $M$ is contained in an injective $R$-module.

## Exercise 3

Let $G$ be an abelian group, written additively. Its **Pontrjagin dual** is the group

$$G^* = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z}).$$

(a) Prove that for any nonzero $a \in G$, there exists a homomorphism $f : G \to \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$. *Why do we care? See remark below.*

(b) Prove that if $0 \to A \to G \to B \to 0$ is an exact sequence of abelian groups, then so is $0 \to B^* \to G^* \to A^* \to 0$.

(c) If $G$ is a finite abelian group, prove that $G^* \cong G$.

**Remark.** In the wild, we usually let $G$ be a locally compact abelian topological group (e.g. $\mathbb{R}^n, \mathbb{C}^\times, \mathbb{Q}_p$, any torus $\mathbb{T}^n$), and the Pontrjagin dual $\widehat{G} = \mathrm{Hom}(G, \mathbb{R}/\mathbb{Z})$ consists of all continuous homomorphisms from $G$ to the circle group $\mathbb{R}/\mathbb{Z}$. This allows us to generalize Fourier transform to any such $G$. In this exercise, we are treating $\mathbb{Q}$ as a topological space with the discrete topology.

## Exercise 4

Let $\varphi : A \to B$ be an $R$-module homomorphism. Prove that $\operatorname{coker}\varphi$ is a pushout. *Hint: Come up with a pushout diagram for* $\operatorname{coker}\varphi$*, and show that* $\operatorname{coker}\varphi$ *satisfies the universal property of pushouts.*

## Exercise 5

In this exercise, you will prove that in the the category ${}_R\mathbf{Mod}$, pullbacks and pushouts always exist.

(a) Let $f : A \to C$ and $g : B \to C$ be two $R$-module maps. Prove that the **pullback** exists, by considering the set

$$P = \{(a, b) \in A \times B \mid f(a) = g(b)\} \qquad (= A \times_C B)$$

Here is a `tikzcd` pullback diagram with some suggested map names:

$$\begin{array}{ccc} P & \xrightarrow{\beta} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

*First, briefly verify that $P$ is actually an object in ${}_R\mathbf{Mod}$; one sentence is enough. Then check that $(P, \alpha, \beta)$ is a solution and that it satisfies the universal property of pullbacks.*

(b) Now let $f : C \to A$ and $g : C \to B$ be two $R$-module maps. Prove that the **pushout** exists, by defining

$$S = \{(f(c), -g(c)) \in A \oplus B \mid c \in C\}$$

and defining the $R$-module

$$Q = (A \oplus B)/S.$$

*First briefly verify that $S$ is actually a submodule of $A \oplus B$; one sentence is enough.*

## Exercise 6

**Definition.** Let $\mathscr{C}$ be a category.

- An object $A \in \mathscr{C}$ is an **initial object** if, for every object $C \in \mathscr{C}$, there exists a unique morphism $A \to C$.
- An object $\Omega \in \mathscr{C}$ is a **terminal object** if, for every object $C \in \mathscr{C}$, there exists a unique morphism $C \to \Omega$.
- An object $Z \in \mathscr{C}$ is a **zero object** if it is both initial and terminal.

(a) Prove that initial and terminal objects are unique (up to unique isomorphism) if they exist.

(b) Suppose $\mathscr{C}$ contains a terminal object $\Omega$. Characterize products in $\mathscr{C}$ as pullbacks.

(c) Suppose a category contains an initial object $A$. Characterize coproducts in $\mathscr{C}$ as pushouts.

(d) What are the initial and terminal objects in **Set**? Prove that **Set** does not have a zero object.

(e) What are the initial and terminal objects in **Ab**? What is the zero object?