HW05

① (a) $\mathbb{Q}$-VS are abelian groups

For $c \in \mathbb{Q}$, $\sum_i a_i \otimes q_i \in A \otimes_{\mathbb{Z}} \mathbb{Q}$,

let $c\left(\sum_i a_i \otimes q_i\right) = \sum_i a_i \otimes c q_i$

Verify this satisfies $\mathbb{Q}$-VS axioms.

(b) $S \otimes_R M \ni sr \otimes m = s \otimes rm$

On the left, sr is multiplication in S.

On the right, rm is defined b/c $M \in {}_R\mathrm{Mod}$.

② $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is an $\mathbb{R}$-module by

$r\left(\sum_i z_i \otimes w_i\right) = \sum_i r z_i \otimes w_i \quad \left(= \sum z_i r \otimes w_i = \sum z_i \otimes r w_i \text{ etc...}\right)$

Similarly for $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$.

However, as $\mathbb{R}$-VSs,

- $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has $\mathbb{R}$-basis $\{1 \otimes 1, 1 \otimes i, i \otimes 1, i \otimes i\}$

  $\Rightarrow$ 4-diml.

- $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C} \Rightarrow$ 2-diml.

  $z \otimes w \longmapsto zw \otimes 1$

③ (a) ✓

(b) $B = \{e_i \otimes e_j\}_{1 \le i,j \le n}$

(Order this basis by lexicographic order, for example)

A is the length $n \times n$ row vector where

the entries corresponding to $e_i \otimes e_i$ are 1,

and the entries corr. to $e_i \otimes e_j = 0$ when $i \neq j$.

④ The appropriate basis for $V \otimes W$ is $\mathcal{B} = \{v_i \otimes w_j\}$.

The ordering is lexicographic:

(top left corner:)

$$\begin{array}{c|ccccc|cccc}
 & v_1 \otimes w_1 & v_1 \otimes w_2 & v_1 \otimes w_3 & \cdots & & v_2 \otimes w_1 & v_2 \otimes w_2 & \cdots & \text{etc} \\
\hline
v_1 \otimes w_1 & & & & & & & & & \\
\vdots & & a_{11} B & & & & & & & \\
v_1 \otimes w_n & & & & & & & & & \\
\hline
 & & & & & & & & &
\end{array}$$

Check that with respect to this (ordered) basis, the matrix for $S \otimes T$ is indeed $A \otimes B$ by verifying that $A \otimes B$ sends the standard basis vector

$$\begin{aligned}
v_i \otimes w_j \longmapsto & \; S v_i \otimes T w_j \\
&= \left(\sum_{k=1}^{m} a_{ki} v_k \right) \otimes \left(\sum_{\ell=1}^{n} b_{\ell j} w_\ell \right) \\
&= \sum_{k=1}^{m} \sum_{\ell=1}^{n} a_{ki} b_{\ell j} \; v_k \otimes w_\ell
\end{aligned}$$

(5)

(a) Note that if R is an integral domain, $R \hookrightarrow Q$ by $r \mapsto \frac{r}{1}$.

This is injective: if $\frac{r}{1} \sim \frac{r'}{1}$, then $\exists\, x \in R^{\times}$ such that $x(r \cdot 1 - 1 \cdot r') = 0$. Since $x \neq 0$, we must have $r = r'$.

Write the elements of the quotient module $Q/R = \{ q + R \mid q \in Q \}$ as $[q]$, eg. $\left[\frac{n}{d}\right] = \frac{n}{d} + R$.

(note that $[0] = 0 + R$ is the additive identity in Q/R.)

Consider any pure tensor $\left[\frac{n_1}{d_1}\right] \otimes \left[\frac{n_2}{d_2}\right] \in Q/R \otimes_R Q/R$.

Since $\frac{n_1}{d_1} = \frac{n_1 d_2}{d_1 d_2} = \frac{n_1}{d_1 d_2} \cdot d_2$, we have

$$\left[\frac{n_1}{d_1}\right] \otimes \left[\frac{n_2}{d_2}\right] = \left[\frac{n_1}{d_1 d_2}\right] \cdot d_2 \otimes \left[\frac{n_2}{d_2}\right] = \left[\frac{n_1}{d_1 d_2}\right] \otimes d_2 \left[\frac{n_2}{d_2}\right]$$

$$= \left[\frac{n_1}{d_1 d_2}\right] \otimes [0] = 0,$$

$\Rightarrow Q/R \otimes_R Q/R$ is the 0 module.

(b) $$\underbrace{\frac{r}{d} \otimes n'}_{\text{arbitrary pure tensor}} = \frac{1}{d} \cdot r \otimes n' = \frac{1}{d} \otimes \underbrace{rn'}_{n}$$

⑥ (a) $d = \gcd(m,n)$

$\Rightarrow m = dm_1,\ n = dn_1$ where $\gcd(m_1,n_1) = 1$.

& also note $d \le m,\ d \le n$.

- Let $\pi_m : \mathbb{Z}/m\mathbb{Z} \twoheadrightarrow \mathbb{Z}/d\mathbb{Z}$, $\pi_n : \mathbb{Z}/n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/d\mathbb{Z}$.

  be the natural maps.

Define $\varphi : \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z}$

to be the composition (of maps in $\mathbb{Z}$-mod)

$$\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \xrightarrow{\pi_m \times \pi_n} \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \xrightarrow{\mu} \mathbb{Z}/d\mathbb{Z}.$$

Check that φ is $\mathbb{Z}$-bilinear.

By U.P., $\exists$ map $\tilde{\varphi} : \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z}$.

$$(\bar{a}, \bar{b}) \longmapsto \overline{ab}$$

- Define $\psi : \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$

$$c \longmapsto c(1 \otimes 1). = \bar{c} \otimes 1 = 1 \otimes \bar{c}$$

Then $\ker\psi$ contains $m\mathbb{Z}$ and $n\mathbb{Z}$, so $\ker\psi$ contains $d\mathbb{Z}$.

$\Rightarrow \psi$ factors through $\mathbb{Z}/d\mathbb{Z}$

$\Rightarrow$ induces a map $\Psi : \mathbb{Z}/d\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$.

$$\bar{c} \longmapsto c(1 \otimes 1)$$

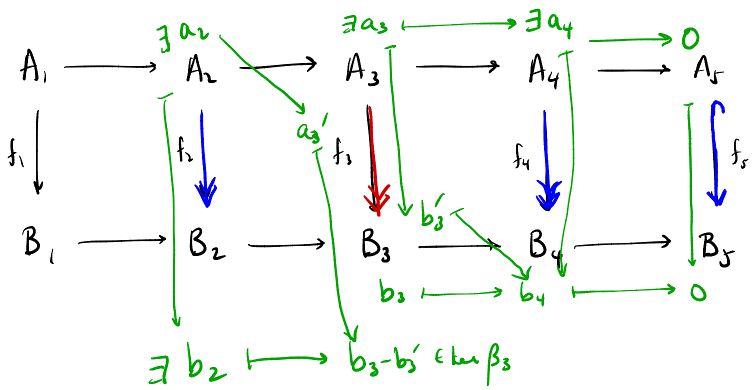
- Check that Ψ and $\tilde{\varphi}$ are inverses.

(b) $(a \otimes b)(a' \otimes b') = aa' \otimes bb' \longmapsto \overline{aa'bb'} = \overline{ab} \cdot \overline{a'b'}$.

7)

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[r] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

(a)



$\Rightarrow f_3(a_3' - a_3)$

etc. :)