MAT 250B HW05

[add your name here]

Due Friday, 2/9/24 at 11:59 pm on Gradescope

Reminder Your homework submission **must be typed** (TeX'ed) up in full sentences, with proper mathematical formatting.

Exercise 1

A very common use of tensor products in the wild is in the process of **base change**.

Let $f : R \to S$ be a ring homomorphism. In ${}_S N$ is a left S-module, then we can also view it as a left R-module ${}_R N$ with

$$rn = f(r)n.$$

We have thus changed the *base ring*, and have performed a *base change*.

If $i : R \hookrightarrow S$ is an inclusion of R as a subring of S, then S is an **extension** of the ring R. In this case, ${}_R N$ is obtained from ${}_S N$ by a **restriction of scalars**.

On the other hand, if ${}_R M$ is a left R-module, in order to change the base ring to S, we **extend the scalars** by tensoring with S over R:

$$S \otimes_R M. \tag{1}$$

The module $S \otimes_R M$ is called the *left S-module obtained by extensions of scalars from the left R-module M*.

(a) Briefly explain why any $\mathbb{Q}$ vector space is also a $\mathbb{Z}$-module. Then let A be an abelian group and prove that $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}$ vector space.

(b) Verify the details of (1) above. That is, verify how an element $r \in R$ acts on both sides of the $\otimes$ symbol.

Exercise 2

Show that $\mathbb{C} \otimes_R \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are both left $\mathbb{R}$-modules but are not isomorphic as $\mathbb{R}$-modules.

Exercise 3

(a) Prove that the usual dot product

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$
$$(a, b) \mapsto \langle a, b \rangle = \sum_{i=1}^{n} a_i b_i$$

is an $\mathbb{R}$-bilinear map.

(b) By the universal property of the tensor product, $\langle \cdot, \cdot \rangle$ descends to an $\mathbb{R}$-linear map

$$\Phi : \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^n \to \mathbb{R}.$$

Build a basis for $\mathbb{R}^n \otimes \mathbb{R}^n$ from the standard basis vectors $e_1, e_2, \ldots, e_n$ for $\mathbb{R}^n$. Then determine the matrix A for Φ with respect to this basis.

Notice that $\Phi \in (\mathbb{R}^n \otimes \mathbb{R}^n)^* = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n \otimes \mathbb{R}^n, \mathbb{R})$. *Therefore A should be a row vector.*

Exercise 4

(How tensors are related to tensor products)

Let V and W be finite-dimensional vector spaces over a field $\mathbb{F}$, with bases $\{v_1, v_2, \ldots, v_m\}$ and $\{w_1, w_2, \ldots, w_n\}$, respectively. Let $S : V \to V$ be a linear transformation with matrix $A = [a_{ij}]$ with respect to the basis $\{v_i\}$. Let $T : W \to W$ be a linear transformation with matrix $B = [b_{ij}]$ with respect to the basis $\{w_i\}$.

Show that the matrix of the linear transformation $S \otimes T : V \otimes W \to V \otimes W$, with respect to a suitable ordering of the basis vectors $v_i \otimes w_j$, is their **Kronecker product**, the $nm \times nm$ matrix written in block form as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix}.$$

Exercise 5

Let R be an integral domain, and let $Q = \mathrm{Frac}(R)$, the quotient field of R. (*Have you seen localization before? If not, we will cover this briefly in class.*)

(a) Prove that $(Q/R) \otimes_R (Q/R) = 0$.

(b) Let N be a left R-module. Prove that every element of the tensor product $Q \otimes_R N$ can be written as a simple (i.e. pure) tensor of the form $(1/d) \otimes n$ for some nonzero $d \in R$ and some $n \in N$.

Exercise 6

Let $m, n \in \mathbb{N}$ (positive integers), and let $d = \gcd(m, n)$.

(a) Prove that there is an isomorphism of abelian groups $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$.

(b) Prove that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ as commutative rings.

Exercise 7

(The Five Lemma) Consider a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

Prove the following statements.

If you know how to use spectral sequences, the proofs can be very quick. If this is the first time you're proving the Five Lemma, you should prove these facts without spectral sequences.

(a) If f_2 and f_4 are surjective and f_5 is injective, then f_3 is surjective.

(b) If f_2 and f_4 are injective and f_1 is surjective, then f_3 is injective.

(c) If f_1 is surjective, f_2 and f_4 are isomorphisms, and f_5 is injective, then f_3 is an isomorphism.

(d) Give an example of a commutative diagram like the above in which no middle map f_3 exists.