HW 06

① If k is commutative, then $(P\otimes Q)\otimes S \cong P\otimes(Q\otimes S)$

Hence $(P\otimes Q)\otimes -$ is naturally isomorphic to $P\otimes(Q\otimes -)$

(i.e. the composition of functors).

Since $Q\otimes -$ is exact and $P\otimes -$ is exact (as P, Q are flat),

we are done.

② (a) Define the $D^{-1}R$ action:

$$\frac{r}{d}\cdot m = \frac{rm}{d}$$

i.e. $(d,r)\cdot(1,m) = (d,rm)$.

Check that this is an action (associative, etc.)

Define $D^{-1}R$-maps

$$\varphi : D^{-1}R\otimes M \longrightarrow D^{-1}M$$
$$\frac{r}{d}\otimes m \longmapsto \frac{rm}{d} \qquad \text{i.e. } (d,r)\otimes m \mapsto (d,rm)$$

$$\psi : D^{-1}M \longrightarrow D^{-1}R\otimes M$$
$$\frac{m}{d} \longmapsto \frac{1}{d}\otimes m \qquad \text{i.e. } (d,m) \longmapsto (d,1)\otimes m$$

Check these are well-defined and are inverses.

(b) $D^{-1}R$ is a flat R-module :

ISTS $D^{-1}R \otimes_R -$ preserves injections, since we already know it's right-exact.

Let $i : A \hookrightarrow B$ be an injective R-map.

Consider

$$\begin{tikzcd} D^{-1}R \otimes A \arrow[r, "1 \otimes f"] \arrow[d, "\tau"] & D^{-1}R \otimes B \arrow[d, "\tau"] \\ D^{-1}A \arrow[r, "D^{-1}f"] & D^{-1}B \\ \frac{a}{d} \arrow[r, mapsto] & \frac{f(a)}{d} \end{tikzcd}$$

Check that τ is a natural isomorphism.

Thus it suffices to check that $D^{-1}f$ is injective.

Suppose $D^{-1}f\left(\frac{a}{d}\right) = 0$. Then $\frac{f(a)}{d} \sim \frac{0}{1}$ ie

$\exists x \in D$ such that $xf(a) = 0$ in B.

$\Rightarrow f(xa) = 0$. Since f is injective, $xa = 0$.

$\Rightarrow \frac{a}{d} = \frac{xa}{xd} = \frac{0}{xd} = 0$.

$\Rightarrow D^{-1}f$ is injective.

Thus $D^{-1}R \otimes -$ is an exact functor, and so $D^{-1}R$ is a flat R-module. This also means that the "localize at D" functor

$$D^{-1}(\cdot) \; : \; R\text{-mod} \longrightarrow D^{-1}R\text{-mod}$$

preserves exact sequences.

③

Suppose we have a natural isom τ from F to G.

(a) Suppose F is left exact and let

$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be exact.

Then $0 \to FA \xrightarrow{F\alpha} FB \xrightarrow{F\beta} FC$ is exact.

This commutes:

$$\begin{tikzcd}
0 \arrow[r] & FA \arrow[r, "F\alpha"] \arrow[d, "\tau_A", "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & FB \arrow[r, "F\beta"] \arrow[d, "\tau_B", "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & FC \arrow[d, "\tau_C", "\cong"'] \\
0 \arrow[r] & GA \arrow[r, "G\alpha"] & GB \arrow[r, "G\beta"] & GC
\end{tikzcd}$$

- $G\alpha$ is injective

  $G\alpha = \tau_B(F\alpha)\tau_A^{-1}$, $F\alpha$ is injective, and τ_B, τ_A^{-1} are isoms.

- $\operatorname{im} G\alpha \subseteq \ker G\beta$:

  $G\beta \cdot G\alpha = \tau_C(F\beta)(F\alpha)\tau_A^{-1} = 0.$

- $\ker G\beta \subseteq \operatorname{im} G\alpha$

  If $b \in GB$ satisfies $G\beta(b) = 0$, then $\tau_C F\beta\, \tau_B^{-1}(b) = 0$

  $\Rightarrow \tau_B^{-1}(b) \in \ker F\beta \Rightarrow \tau_B^{-1}(b) \in \operatorname{im} F\alpha$

  $\Rightarrow \exists\ a \in FA$ s.t. $F\alpha(a) = \tau_B^{-1}(b)$

  $\Rightarrow G\alpha(\tau_A(a)) = b$, where $\tau_A(a) \in GA$.

(b) just check surjective map

(c) quick conclusion.

④

(a) Let $M, N \in (R,S)$-bimod

and let $\varphi : M \longrightarrow N$ be an (R,S)-map

$$
\begin{tikzcd}
{[f : A\otimes N \to C]} \arrow[r, mapsto] \arrow[ddd, bend right, mapsto] & {[f\circ(\mathrm{id}_A\otimes\varphi) : A\otimes M \to C]} \arrow[ddd, bend left, mapsto, "?"] \\
\mathrm{Hom}_S(A\otimes N, C) \arrow[r] \arrow[d, "\tau_{A,M,C}"'] & \mathrm{Hom}_S(A\otimes M, C) \arrow[d, "\tau_{A,N,C}"] \\
\mathrm{Hom}_R(A, \mathrm{Hom}_S(N, C)) \arrow[r] & \mathrm{Hom}_R(A, \mathrm{Hom}_S(M, C)) \\
\begin{pmatrix} f_a : N \to C \\ n \mapsto f(a\otimes n)\end{pmatrix}_{a\in A} \arrow[r, mapsto] & \begin{pmatrix} f_a\varphi : M \to C \\ m \mapsto f(a\otimes\varphi(m))\end{pmatrix}_{a\in A}
\end{tikzcd}
$$

Check the red arrow sends

$$f\circ(\mathrm{id}_A\otimes\varphi) \longmapsto (f_a\varphi)_{a\in A}$$

Indeed,

$$
\begin{aligned}
\big(f\circ(\mathrm{id}_A\otimes\varphi)\big)_a : \quad & M \longrightarrow C \\
& m \longmapsto \big(f\circ(\mathrm{id}\otimes\varphi)\big)(a\otimes m) \\
& \qquad = f(a\otimes\varphi(m))
\end{aligned}
$$

(b) and (c) are similar. We set up the square to check commutativity for on the next page.

Diagram for (b): $\varphi : M \longrightarrow N$ in Mod_R

$$
\begin{tikzcd}
f \arrow[rr, mapsto] & & f\circ(\varphi\otimes \mathrm{id}_B) \\
\mathrm{Hom}_S(N\otimes B, C) \arrow[rr] \arrow[d, "\tau"] & & \mathrm{Hom}_S(M\otimes B, C) \arrow[d, "\tau"] \\
\mathrm{Hom}_S(N, \mathrm{Hom}_R(B,C)) \arrow[rr] & & \mathrm{Hom}_R(M, \mathrm{Hom}_S(B,C)) \\
\left(\begin{array}{l} f_n : B \to C \\ b \mapsto f(n\otimes b)\end{array}\right)_{n\in N} \arrow[rr, mapsto] & & \left(\begin{array}{l} g_m : B \longrightarrow C \\ b \longmapsto f(\varphi(m)\otimes b)\end{array}\right)_{m\in M}
\end{tikzcd}
$$

?

Diagram for (c): $\varphi : M \longrightarrow N$ in Mod_S

$$
\begin{tikzcd}
f \arrow[rr, mapsto] & & \varphi f \\
\mathrm{Hom}_S(A\otimes_R B, M) \arrow[rr] \arrow[d] & & \mathrm{Hom}_S(A\otimes_R B, N) \arrow[d] \\
\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, M)) \arrow[rr] & & \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, N)) \\
\left(\begin{array}{l} f_a : B \longrightarrow M \\ b \longmapsto f(a\otimes b)\end{array}\right)_{a\in A} \arrow[rr, mapsto] & & \left(\begin{array}{l} \varphi\circ f_a : B \longrightarrow N \\ b \longmapsto \varphi f(a\otimes b)\end{array}\right)_{a\in A}
\end{tikzcd}
$$

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