

HW 06

① If k is commutative, then $(P \otimes Q) \otimes S \cong P \otimes (Q \otimes S)$

Hence $(P \otimes Q) \otimes -$ is naturally isomorphic to $P \otimes (Q \otimes -)$

(i.e. the composition of functors).

Since $Q \otimes -$ is exact and $P \otimes -$ is exact (as P, Q are flat), we are done.

② (a) Define the $D^{-1}R$ action:

$$\frac{r}{d} \cdot m = \frac{rm}{d}$$

$$\text{i.e. } (d, r) \cdot (1, m) = (d, rm).$$

Check that this is an action (associative, etc.)

Define $D^{-1}R$ -maps

$$\varphi: D^{-1}R \otimes M \longrightarrow D^{-1}M$$

$$\frac{r}{d} \otimes m \longmapsto \frac{rm}{d}$$

$$\text{i.e. } (d, r) \otimes m \longmapsto (d, rm)$$

$$\psi: D^{-1}M \longrightarrow D^{-1}R \otimes M$$

$$\frac{m}{d} \longmapsto \frac{1}{d} \otimes m$$

$$\text{i.e. } (d, m) \longmapsto (d, 1) \otimes m$$

Check these are well-defined and are inverses.

(b) $D^{-1}R$ is a flat R -module :

ISTS $D^{-1}R \otimes_R -$ preserves injections, since we already know it's right-exact.

Let $i: A \hookrightarrow B$ be an injective R -map.

Consider

$$\begin{array}{ccc}
 D^{-1}R \otimes A & \xrightarrow{1 \otimes f} & D^{-1}R \otimes B \\
 \downarrow \tau & & \downarrow \tau \\
 D^{-1}A & \xrightarrow{D^{-1}f} & D^{-1}B \\
 \frac{a}{d} & \xrightarrow{\quad\quad\quad} & \frac{f(a)}{d}
 \end{array}$$

Check that τ is a natural isomorphism.

Thus it suffices to check that $D^{-1}f$ is injective.

Suppose $D^{-1}f\left(\frac{a}{d}\right) = 0$. Then $\frac{f(a)}{d} \sim \frac{0}{1}$ i.e.

$\exists x \in D$ such that $xf(a) = 0$ in B .

$\Rightarrow f(xa) = 0$. Since f is injective, $xa = 0$.

$\Rightarrow \frac{a}{d} = \frac{xa}{xd} = \frac{0}{xd} = 0$.

$\Rightarrow D^{-1}f$ is injective.

Thus $D^{-1}R \otimes -$ is an exact functor, and so $D^{-1}R$ is a flat R -module. This also means that the "localize at D " functor

$$D^{-1}(\cdot) : R\text{-mod} \longrightarrow D^{-1}R\text{-mod}$$

preserves exact sequences.

③

Suppose we have a natural isom τ from F to G .

(a) Suppose F is left exact and let

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \quad \text{be exact.}$$

Then $0 \rightarrow FA \xrightarrow{F\alpha} FB \xrightarrow{F\beta} FC$ is exact.

This commutes:

$$\begin{array}{ccccccc} 0 & \rightarrow & FA & \xrightarrow{F\alpha} & FB & \xrightarrow{F\beta} & FC \\ & & \cong \downarrow \tau_A & \circlearrowright & \cong \downarrow \tau_B & \circlearrowright & \cong \downarrow \tau_C \\ 0 & \rightarrow & GA & \xrightarrow{G\alpha} & GB & \xrightarrow{G\beta} & GC \end{array}$$

• $G\alpha$ is injective

$G\alpha = \tau_B(F\alpha)\tau_A^{-1}$, $F\alpha$ is injective, and τ_B, τ_A^{-1} are isoms.

• $\text{im } G\alpha \subseteq \text{ker } G\beta$:

$$G\beta \cdot G\alpha = \tau_C(F\beta)(F\alpha)\tau_A^{-1} = 0.$$

• $\text{ker } G\beta \subseteq \text{im } G\alpha$

If $b \in GB$ satisfies $G\beta(b) = 0$, then $\tau_C F\beta \tau_B^{-1}(b) = 0$

$$\Rightarrow \tau_B^{-1}(b) \in \text{ker } F\beta \Rightarrow \tau_B^{-1}(b) \in \text{im } F\alpha$$

$$\Rightarrow \exists a \in FA \text{ s.t. } F\alpha(a) = \tau_B^{-1}(b)$$

$$\Rightarrow G\alpha(\tau_A(a)) = b, \text{ where } \tau_A(a) \in GA.$$

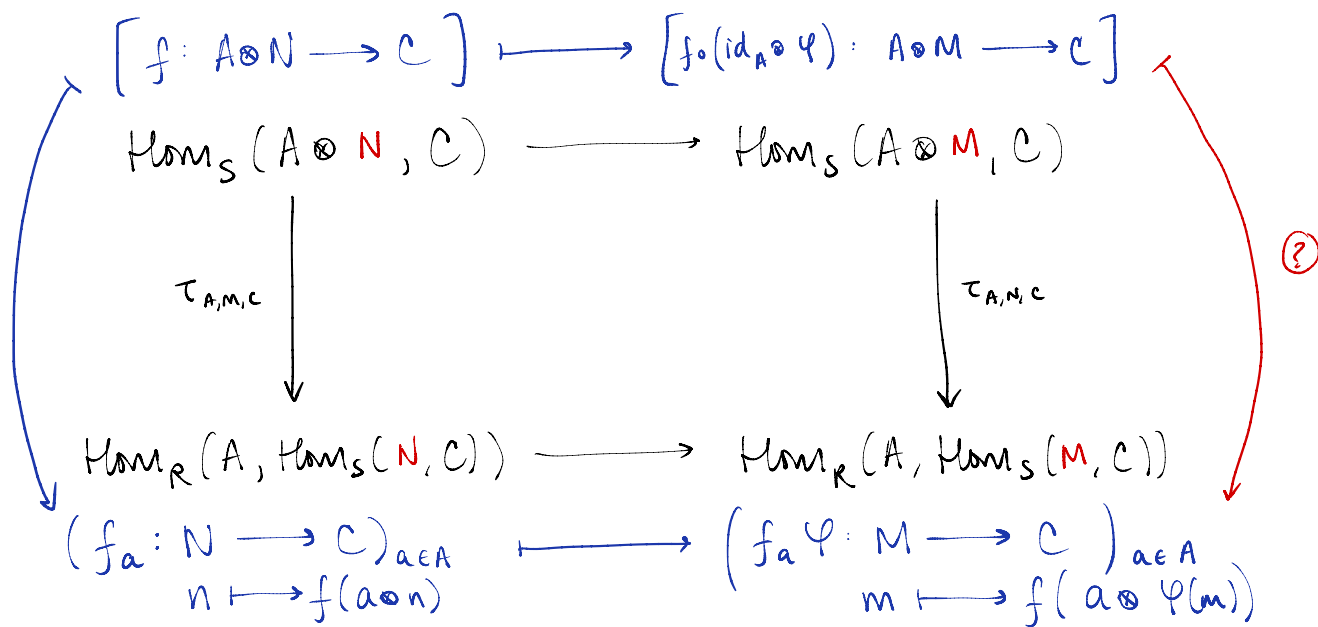
(b) just check surjective map

(c) quick conclusion.

④

(a) Let $M, N \in (R, S)$ -bimod

and let $\varphi: M \rightarrow N$ be an (R, S) -map



Check the red arrow sends

$$f \circ (\text{id}_A \otimes \varphi) \xrightarrow{\quad} (f_a \varphi)_{a \in A}$$

Indeed,

$$\begin{aligned}
 (f \circ (\text{id}_A \otimes \varphi))_a : M &\longrightarrow C \\
 m &\longmapsto (f \circ (\text{id}_A \otimes \varphi))(a \otimes m) \\
 &= f(a \otimes \varphi(m))
 \end{aligned}$$

(b) and (c) are similar. we set up the square to check commutativity for on the next page.

Diagram for (b): $\varphi: M \rightarrow N$ in Mod_R

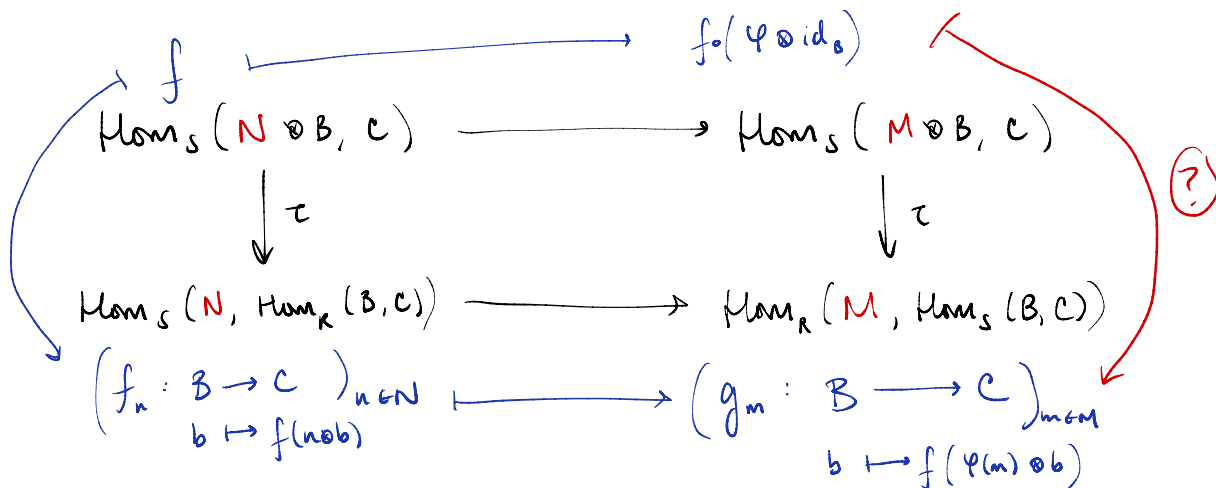


Diagram for (c): $\varphi: M \rightarrow N$ in Mod_S

