MAT 250B HW06

[add your name here]

Due Friday, 2/16/24 at 11:59 pm on Gradescope

Reminder Your homework submission **must be typed** (TeX'ed) up in full sentences, with proper mathematical formatting.

Midterm Exam The midterm exam covers material from

- Lectures 01–11
- HW01–HW05
- §6.1 – §6.6 in the text

This will be a 50-minute pen-and-paper exam. As such, I will test you more on your understanding of definitions, properties, and common proof techniques; the proofs will not require a 'spark of genius', but rather familiarity with the types of proofs we have already seen in class and on homeworks.

Exercise 1

Let k be a commutative ring, and let P, Q be flat k-modules. Prove that $P \otimes_k Q$ is also a flat k-module.

Exercise 2

(Localization is exact) Let R be a commutative ring, and let $D^{-1}R$ be its localization at a multiplicatively closed subset $D \subset R$ (containing 1). Let $M \in R - \mathbf{Mod}$.

Let $D^{-1}M$ denote the $D^{-1}R$-module defined by

$$D^{-1}M = (D \times M)/\sim \qquad (d, m) \sim (e, m) \text{ iff } x(dn - em) = 0 \text{ for some } x \in D.$$

(a) Verify that $D^{-1}M$ is indeed a $D^{-1}R$-module. Then verify that $D^{-1}M \cong D^{-1}R \otimes_R M$.

(b) Prove that $D^{-1}R$ is flat. Conclude that localization of modules is an exact functor. *That is, clarify the meaning of the mantra "localization is exact".*

Exercise 3

Here is a fact we've been using in class quite often.

Let $F, G : {}_R\mathbf{Mod} \to \mathbf{Ab}$ be two covariant additive functors. Suppose that F and G are naturally isomorphic.

(a) Prove that F is left exact iff G is left exact. [1]

(b) Prove that F is right exact iff G is right exact.

(c) Conclude that F is exact iff G is exact.

Exercise 4

Recall that we have two adjoint isomorphisms.

- For **right modules** $A_R \in \mathbf{Mod}_R, C_S \in \mathbf{Mod}_S$: Let ${}_RB_S \in (R,S)$-bimod. Then there is an isomorphism of abelian groups

$$\tau_{A,B,C} : \mathrm{Hom}_S(A \otimes_R B, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,C)).$$

- For **left modules** ${}_RA \in_R \mathbf{Mod}, {}_SC \in_S \mathbf{Mod}$: Let ${}_SB_R \in (S,R)$-bimod. Then there is an isomorphism of abelian groups

$$\tau'_{A,B,C} : \mathrm{Hom}_S(B \otimes_R A, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,C)).$$

In this exercise, you will check that the maps we obtain from $\tau_{A,B,C}$ by fixing two out of the three in $\{A,B,C\}$ are in fact natural isomorphisms between functors. *The proofs for the adjoint isomorphism of left modules is similar.*

(a) Prove that (the collection of maps, one for each pair of choices A, C)

$$\tau_{A,-,C} : \mathrm{Hom}_S(A \otimes_R -, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(-,C))$$

is a natural isomorphism.

Remark. This proves that

$$F = A \otimes_R - \qquad \text{and} \qquad G = \mathrm{Hom}_S(-,C) = \mathrm{Hom}_{\mathbf{Mod}_S}(-,C)$$

are an adjoint pair of functors ${}_R\mathbf{Mod} \to \mathbf{Ab}$.

The analogous proof for left modules shows that

$$F = - \otimes_R A \qquad \text{and} \qquad G = \mathrm{Hom}_S(-,C) = \mathrm{Hom}_{{}_S\mathbf{Mod}}(-,C)$$

are an adjoint pair of functors $\mathbf{Mod}_R \to \mathbf{Ab}$.

(b) Prove that (the collection of maps, one for each pair of choices B, C)

$$\tau_{-,B,C} : \mathrm{Hom}_S(- \otimes_R B, C) \to \mathrm{Hom}_R(-, \mathrm{Hom}_S(B,C))$$

is a natural isomorphism.

(c) Prove that (the collection of maps, one for each pair of choices A, B)

$$\tau_{A,B,-} : \mathrm{Hom}_S(A \otimes_R B, -) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,-))$$

is a natural isomorphism.

[1] I don't think I need to tell you this, but just in case: you only need to prove one direction, since F and G are arbitrary.