HWD8

① (a) Check bilinearity; clear because scalar matrices commute with all matrices (under mult.).

ISTS $Tr(AB) = Tr(BA)$.

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{21}b_{12} & * \\ * & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

$\Rightarrow Tr(AB) = (a_{11}b_{11} + a_{21}b_{12}) + (a_{21}b_{12} + a_{22}b_{22})$

Similarly, $Tr(BA) = (b_{11}a_{11} + b_{21}a_{12}) + (b_{21}a_{12} + b_{22}a_{22})$

(by swapping a's for b's.)

<u>Rmk.</u> In general, in $Mat_{n\times n}(\mathbb{R})$, $Tr(AB) = Tr(BA)$. This is a defining properties of functions called "traces".

(b) Direct computation. Notice that $e_{ij}e_{kl} = \begin{cases} e_{il} & \text{if } j=k \\ 0 & \text{if } j \neq k. \end{cases}$

So we have a multiplication table for e_ie_j:

(A) e_i \ e_j (B)	e_{11}	e_{12}	e_{21}	e_{22}
e_{11}	e_{11}	e_{12}	0	0
e_{12}	0	0	e_{11}	e_{12}
e_{21}	e_{21}	e_{22}	0	0
e_{22}	0	0	e_{21}	e_{22}

Take trace of each cell:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{inner product matrix} =: S$$

(c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{matrix} tr = 0 \\ det = -1 \end{matrix}$ $\Rightarrow \lambda = \pm 1. \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\Rightarrow S \sim \left[\begin{array}{c|c} I_3 & \\ \hline & -I_1 \end{array}\right]$. $\Rightarrow$ rank 4, signature 2.

② (a) From calculus we know

$\int (f_1+f_2)g\,dx = \int f_1 g\,dx + \int f_2 g\,dx$ (and same for f and g_1+g_2)

and $c\int fg\,dx = \int (cf)g\,dx = \int f(cg)\,dx$.

Use these properties to argue.

(b) Symmetry: multiplication of real numbers is commutative, so $fg = gf$ (at every point $x \in [-1,1]$).

Nondegenerate: We will show that if $f \neq 0$, then there is some g (namely, $g=f$) such that $\langle f,g\rangle \neq 0$.

Suppose $f \neq 0$. Then $(f(a))^2 > 0$ at some $a \in [-1,1]$.

In fact, we may choose $a \in (-1,1)$ (by continuity).

Then there is some interval $(a-\varepsilon, a+\varepsilon)$ on which $(f(x))^2 > 0$. Then since $(f(x))^2 \geq 0$ for all $x \in [-1,1]$,

$$\int_{-1}^{1} (f(x))^2\,dx \geq \int_{a-\varepsilon}^{a+\varepsilon} (f(x))^2\,dx > 0.$$

$\Rightarrow \langle f,f\rangle \neq 0$.

(c) $V = \operatorname{Span}\{1, x, x^2\}$. Let $(b_0^{(0)}, b_1^{(0)}, b_2^{(0)}) := (1, x, x^2)$.

With resp. to this basis, the inner product matrix is

$$\begin{array}{c|c} & \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{array}{c} 1 \\ x \\ x^2 \end{array} & \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix} \end{array}$$

$$\begin{pmatrix} 2 & 2/3 \\ 2/3 & 2/5 \end{pmatrix} \xrightarrow{r_2' = r_2 - \frac{1}{3}r_1} \begin{pmatrix} 2 & 2/3 \\ 0 & \frac{2}{5}-\frac{2}{9} \end{pmatrix} \xrightarrow{c_2' = c_2 - \frac{1}{3}c_1} \begin{pmatrix} 2 & 0 \\ 0 & \frac{2}{5}-\frac{2}{9} \end{pmatrix}$$

Let $b_0^{(1)} = 1$, $b_1^{(1)} = x$, $b_2^{(1)} = x^2 - \frac{1}{3}$.

With our new basis $\{b_0^{(1)}, b_1^{(1)}, b_2^{(1)}\}$ our matrix is diagonal:

$$\begin{pmatrix} 2 & & \\ & 2/3 & \\ & & 2/5 - 2/9 \end{pmatrix}$$

ie $\langle b_0^{(1)}, b_0^{(1)} \rangle = 2$

Now let $b_0^{(2)} = \frac{1}{\sqrt{2}} b_0^{(1)}$. (ie. $b_0^{(2)} = \frac{1}{\sqrt{2}} \cdot 1$).

Then $\langle b_0^{(2)}, b_0^{(2)} \rangle = \frac{1}{2} \langle b_0^{(1)}, b_0^{(1)} \rangle = 1$

Similarly rescale the others:

$$b_1^{(2)} = \sqrt{\tfrac{3}{2}}\, b_1^{(1)} = \sqrt{\tfrac{3}{2}} \cdot x$$

$\frac{2}{5} - \frac{2}{9} = \frac{18}{45} - \frac{10}{45} = \frac{8}{45}$. $\sqrt{\frac{45}{8}} = \frac{3\sqrt{5}}{2\sqrt{2}}$ $\quad b_2^{(2)} = \frac{3\sqrt{5}}{2\sqrt{2}}\, b_2^{(1)} = \frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{3})$

Then wrt. the basis $\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} \cdot x, \frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{3})\}$,

the inner product matrix is $\begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$.

③ (a) We know $m \wedge n = (-1)\, n \wedge m$.

To move m past all the k m_i factors, we pick up a sign of $(-1)^k$.

(b) Let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis for F as a free R-mod.

The $n \cdot i$ pure tensors $\{b_{j_1} \otimes b_{j_2} \otimes \cdots \otimes b_{j_i}\} =: B^{\otimes i}$ where each $b_{j_\ell} \in B$ generate $\bigotimes^i F$.

Now in $\Lambda(F)$, only the (cosets of) pure tensors where all b_{j_ℓ} are distinct are nonzero. If one pure tensor is a permutation of another, then their cosets in $\Lambda(F)$ are (± 1)-multiples of each other. So there are at most $\binom{n}{i}$ generators:

$\{ b_{j_1} \wedge b_{j_2} \wedge \cdots \wedge b_{j_i} \mid j_1 < j_2 < \cdots < j_i \} =: \mathcal{B}$.

It remains to show that $\mathcal{B}$ is linearly independent.

Let I denote a multiindex of size i, ie $I \subset \{1, \dots, n\} =: [n]$ where $|I| = i$. Let b_I denote the correspondence pure wedge product in $\mathcal{B}$. Let $\widetilde{b_I}$ denote the pure tensor representative of b_I in $T(F)$ where the indices of $b_{j_1} \otimes b_{j_2} \otimes \cdots \otimes b_{j_i}$ are strictly increasing.

Suppose $\sum_{I \subset [n]} r_I b_I = 0$ in $\Lambda^i(F)$.

Then $\sum r_I \widetilde{b_I} \in J$ (recall: $T(F)/J = \Lambda(F)$)

But every pure tensor appearing in an element of J contains a repeated tensor factor, while every $\widetilde{b_I}$ does not. So since $B^{\otimes i}$ is linearly independent, $r_I = 0 \ \forall\ I$. Therefore $\mathcal{B}$ is linearly independent.