MAT 250B HW08

[add your name here]

Due Friday, 3/1/24 at 11:59 pm on Gradescope

Exercise 1

Let $V = \mathrm{Mat}_{2\times 2}(\mathbb{R})$. Define a form on V by

$$\langle A, B\rangle = \mathrm{Tr}(AB)$$

where Tr is the trace.

(a) Show that this is a symmetric bilinear form.

(b) Find the inner product matrix with respect to the standard basis

$$\left\{ e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(c) Determine the rank and signature of this form.

Exercise 2

Let V be the real vector space consisting of real polynomials of degree $\leq N$. Define a form on V by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

(a) Show that $\langle \cdot, \cdot \rangle$ is a bilinear form on V.

(b) Prove that $\langle \cdot, \cdot \rangle$ is symmetric and non-degenerate.

(c) For $N = 2$, find a basis for V so that the inner product matrix is a diagonal matrix with ± 1 along the diagonal.

Exercise 3

Let R be a commutative ring.

(a) Let m and n_i $(1 \leq i \leq k)$ be elements of an R-module M. Prove that

$$m \wedge n_1 \wedge n_2 \wedge \cdots \wedge n_k = (-1)^k n_1 \wedge n_2 \wedge \cdots \wedge n_k \wedge m.$$

(b) Prove that if F is a free R-module of rank n, then $\bigwedge^i(F)$ is a free R-module of rank $\binom{n}{i}$.