MAT 250B Winter 2024

Instructor: Melissa Zhang

Final Exam

By providing my signature below I acknowledge that I abide by the University's academic honesty policy. This is my work, and I did not get any help from anyone else:

Name (sign): ______________________ Name (print): ______________________

Policies

- This is a **closed-book** exam. You may not use the textbook, cheat sheets, notes, or any other outside material. No calculators, computers, phones, or any other electronics are allowed.
- You have **2 hours** to complete this exam. If you are done early, you may leave after handing in your exam packet.
- Everyone must work on their own exam. Any suspicions of collaboration, copying, or otherwise violating the Student Code of Conduct will be forwarded to the Student Judicial Board.

Instructions

1. Use scratch paper to work out the problems.
2. Write down your solutions on the provided paper, **clearly marking where your solutions for each question begins**. Please write legibly and coherently; otherwise style points may be deducted.
3. Submit this exam cover sheet and your solutions to me, in order, so that I can paper clip them. **Do not submit any scratchwork.**

1. (20 points) Let F be a free $\mathbb{Z}$-module. Prove that $\bigcap_{n=1}^{\infty} nF = 0$.

2. (40 points) Consider the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}$.

   (a) Prove that $\mathbb{Q} \oplus \mathbb{Z}$ is flat.

   (b) Prove that $\mathbb{Q} \oplus \mathbb{Z}$ is neither projective nor injective.
   Hint: You may use the result of Question 1.

3. (20 points) Let A be an invertible, real, skew symmetric matrix. Prove that A^2 is symmetric and negative definite.

4. (20 points) Let $m = p^k$ (p prime, $k \in \mathbb{N}$), and suppose $d \mid m - 1$. Prove that the cyclotomic polynomial $\Phi_d(x)$ has $\phi(d)$ roots in $\mathbb{F}_m$.

   Recall that ϕ is Euler's totient function: $\phi(d) = \#\{1 \leq k \leq d \ : \ \gcd(k, d) = 1\}$.

5. (40 points) Let $\alpha = \sqrt{1 + \sqrt{2}} \in \mathbb{C}$.

   (a) Determine the minimal polynomial of α over $\mathbb{Q}$.

   (b) Determine the Galois closure of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

6. (60 points) Let $f(x) = x^4 + 1$.

   (a) Factor $f(x)$ into irreducibles in $\mathbb{Z}[x]$ and $\mathbb{F}_3[x]$.

   (b) Determine the Galois group of $f(x)$ over $\mathbb{Q}$.

   (c) Determine the Galois group of $f(x)$ over $\mathbb{F}_3$.