MAT 250B Winter 2024
Instructor: Melissa Zhang
Midterm Exam

By providing my signature below I acknowledge that I abide by the University's academic honesty policy. This is my work, and I did not get any help from anyone else:

Name (sign): ______________________ Name (print): ______________________

Question	Points	Score
Q1	30	
Q2	40	
Q3	30	
Total:	100	

- This is a **closed-book** exam. You may not use the textbook, cheat sheets, notes, or any other outside material. No calculators, computers, phones, or any other electronics are allowed.
- You have **45 minutes** to complete this exam. If you are done early, you may leave after handing in your exam packet.
- Everyone must work on their own exam. Any suspicions of collaboration, copying, or otherwise violating the Student Code of Conduct will be forwarded to the Student Judicial Board.

Instructions

1. Use scratch paper to work out the problems.
2. Write down your solutions on the provided paper, clearly marking where your solutions for each question begins. Please write legibly and coherently; otherwise style points may be deducted.
3. Submit this exam cover sheet and your solutions to me, in order, so that I can paper clip them. Place your scratch paper in the pile next to me.

1. (30 points) Let S^1 denote the *circle group*, the subgroup of the multiplicative group $\mathbb{C}^\times = \mathbb{C} - \{0\}$ consisting of the complex numbers of modulus 1:

$$S^1 = \{e^{i\theta} \in \mathbb{C}^\times \ : \ \theta \in \mathbb{R}\}.$$

Prove that S^1 is an injective $\mathbb{Z}$-module.

2. (40 points) Let k be a commutative ring.

Lemma. Let $\{M_i\}_{i\in I}$ be family of k-modules, and let N be k-module. Then

$$\left(\bigoplus_{i\in I} M_i\right) \otimes_k N \cong \bigoplus_{i\in I} (M_i \otimes_k N)$$

as k-modules.

Proof. This can be proven by exhibiting an isomorphism φ of k-modules that sends $(\sum_i m_i)\otimes n \mapsto \sum_i (m_i \otimes n)$. $\square$

Also recall that since k is commutative, $M \otimes N \cong N \otimes M$ for k-modules M and N.

(a) (20 points) Using the lemma, prove that the tensor product of two free k-modules is free.

(b) (20 points) Let P_1 and P_2 be projective k-modules. Prove that $P_1 \otimes_k P_2$ is also a projective k-module.

3. (30 points) Consider the following **pushout** diagram of left R-modules and R-maps:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\gamma"] \\ B \arrow[r, "\beta"'] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Prove that if f is injective, then γ is injective.