## MAT 250B Winter 2024 Instructor: Melissa Zhang Midterm Exam

By providing my signature below I acknowledge that I abide by the University's academic honesty policy. This is my work, and I did not get any help from anyone else:

Name (sign): \_\_\_\_\_\_ Name (print): \_\_\_\_\_

Question	Points	Score
Q1	30	
Q2	40	
Q3	30	
Total:	100	

- This is a **closed-book** exam. You may not use the textbook, cheat sheets, notes, or any other outside material. No calculators, computers, phones, or any other electronics are allowed.
- You have 45 minutes to complete this exam. If you are done early, you may leave after handing in your exam packet.
- Everyone must work on their own exam. Any suspicions of collaboration, copying, or otherwise violating the Student Code of Conduct will be forwarded to the Student Judicial Board.

## Instructions

- 1. Use scratch paper to work out the problems.
- 2. Write down your solutions on the provided paper, clearly marking where your solutions for each question begins. Please write legibly and coherently; otherwise style points may be deducted.
- 3. Submit this exam cover sheet and your solutions to me, in order, so that I can paper clip them. Place your scratch paper in the pile next to me.

1. (30 points) Let  $S^1$  denote the *circle group*, the subgroup of the multiplicative group  $\mathbb{C}^{\times} = \mathbb{C} - \{0\}$  consisting of the complex numbers of modulus 1:

$$S^1 = \{ e^{i\theta} \in \mathbb{C}^{\times} : \theta \in \mathbb{R} \}.$$

Prove that  $S^1$  is an injective  $\mathbb{Z}$ -module.

2. (40 points) Let k be a commutative ring.

**Lemma.** Let  $\{M_i\}_{i \in I}$  be family of k-modules, and let N be k-module. Then

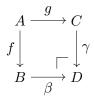
$$\left(\bigoplus_{i\in I} M_i\right)\otimes_k N\cong \bigoplus_{i\in I} \left(M_i\otimes_k N\right)$$

as k-modules.

*Proof.* This can be proven by exhibiting an isomorphism  $\varphi$  of k-modules that sends  $(\sum_i m_i) \otimes n \mapsto \sum_i (m_i \otimes n)$ .

Also recall that since k is commutative,  $M \otimes N \cong N \otimes M$  for k-modules M and N.

- (a) (20 points) Using the lemma, prove that the tensor product of two free k-modules is free.
- (b) (20 points) Let  $P_1$  and  $P_2$  be projective k-modules. Prove that  $P_1 \otimes_k P_2$  is also a projective k-module.
- 3. (30 points) Consider the following **pushout** diagram of left *R*-modules and *R*-maps:



Prove that if f is injective, then  $\gamma$  is injective.