NOTES ON KHOVANOV HOMOLOGY

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Contents

1. Introduction

These notes will be changing during the course. They will likely start out quite informal and become more formal with iteration.

The goal is to take a journey through areas in/around Khovanov homology, to see some open problems, and also to see the algebraic and topological tools in action. For me, Khovanov homology has been a great way to learn a little bit from a lot of different areas, including homological algebra, algebraic geometry, symplectic geometry, homotopy theory, and of course low-dimensional topology.

1.1. **Master list of topics.** Here's a list of topics. We will not get to all of these; this is just a master list, which will help guide the pace of our lectures and also provide potential final project ideas.

- Jones polynomial to Khovanov homology
- TQFT, Frobenius algebra, Bar-Natan category, functoriality
- computation*; torsion*
- deformations, web-foam generalizations
- Rasmussen-Lee spectral sequence, Karoubi envelope; Piccirillo and the Conway knot
- ribbon cobordisms
- reduced theory, relation to gauge/Floer theories, knot detection
- odd khovanov homology; Szabó geometric spectral sequence
- tangles and representations (Khovanov, Chen-Khovanov)

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- annular, Legendrian, and transverse topology applications
- symplectic Khovanov homology; Cautis-Kamnizer, Anno, Anno-Nandakumar
- physical interpretations* (Witten, Gukov, Aganagic)
- spectrification (Lipshitz-Sarkar, Lawson-Lipshitz-Sarkar, Hu-Kriz-Kriz*)
- skein lasagna modules (Morrison-Walker-Wedrich, Manolescu-Neithalath)
- immersed curves (Kotelskiy-Watson-Zibrowius)

* Topics marked with an asterisk (*) are those that I will almost definitely not be lecturing about, or will only touch upon briefly.

Aside 1.1.1. This list has so far been populated from the following sources:

(1) the topics I personally feel are fundamental
(2) Khovanov-Lipshitz's recent survey [KL23]
(3) chatting with you all

Remark 1.1.2. Turner's *A Hitchhiker's Guide to Khovanov Homology* contains exposition for a few of the topics above. (add: citation)

2. Knots and Topology

2.1. **Knots and links.**

If you want to learn more about knots in S^3, Rolfsen is a great resource. There is also a little green book by Livingston that also seems really useful. (add: references)

A knot is sometimes defined as a smooth embedding $S^1 \hookrightarrow S^3$.

Notice that we can

- reparametrize the embedding, preserving the image setwise
- perform an *ambient isotopy* on the knot ('isotop' the knot)

Definition 2.1.1. Let $K, K' : S^1 \hookrightarrow S^3$ be two smooth embeddings. We say K and K' are (smoothly, ambiently) *isotopic* if there exists a smooth family of diffeomorphisms

$$\{\varphi_t : S^3 \to S^3\}_{t\in[0,1]}$$

such that $\varphi_0 = \mathrm{id}$ and $\varphi(1) \circ K = K'$.

Remark 2.1.2. Convince yourself of the following:

(1) "Ambiently isotopic" is an equivalence relation. The equivalence classes are called *isotopy classes* of knots.
(2) In other words, an ambient isotopy smoothly morphs the embedding K into the embedding K', through a family of diffeomorphisms of S^3.
(3) An order-preserving reparametrization is an ambient isotopy.

Remark 2.1.3. In practice, when I say *knot*, I'm probably referring to either (1) the image of the knot in S^3 or (2) an entire isotopy class.

Example 2.1.4. 'The' *unknot* U is the equivalence class of the standard embedding $S^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow S^3$.

Definition 2.1.5. Any diffeomorphic copy of S^1 can have two possible orientations. The *orientation* of a knot is given by the direction $\frac{dK}{d\theta}$.[1]

Definition 2.1.6. The *reverse* K^r (also denoted $\bar{K}$) of a knot K is the[2] knot obtained by precomposing K with an order-reversing diffeomorphism $\rho : [0,1] \to [0,1]$.

Let τ be an orientation-reversing diffeomorphism of S^3. The *mirror* of a knot K is the knot $m(K) = \tau \circ K$.

Remark 2.1.7. An *unoriented knot* is just the knot after you forget about the orientation. You can think of this as the union of the isotopy classes of K and $\bar{K}$.

[1] This is terrible notation and should not ever appear again, because of how we use the term 'knot'; see Remark 2.1.3.

[2] By using the article 'the', I'm using the term 'knot' in the sense of Remark 2.1.3 (2).

Exercise 2.1.8. First observe that the orientation of S^3 does not reverse under isotopy.

Prove that

- In general, $m(K)$ is not necessarily isotopic to K, even if we treat them as unoriented knots. You can prove this via a (counter)example. Proving this directly is hard. Use the Jones polynomial, introduced in Section 2.3.
- In general, K^r is not necessarily isotopic to K. There is no general algorithm for determining when $K \not\sim K^r$! For this problem, do a search online and see if you can an example in the literature, as well as the relevant terms for describing this kind of symmetry.

Then, find examples of knots K that do happen to satisfy the following, and exhibit an explicit isotopy:

- $K \sim K^r$
- $K \sim m(K)$

Diagrammatically exhibit the isotopy by applying Reidemeister moves; see Section 2.2. If there's an obvious part of the isotopy that's easy to describe but takes a lot of Reidemeister moves, you can just note what you're doing between two pictures.

Exercise 2.1.9. (Important) A *link* with $\ell \in \mathbb{Z}_{\geq 0}$ components is a smooth embedding

$$L : \coprod_{i=1}^{\ell} S^1 \hookrightarrow S^3$$

The link with zero components is called the *empty link*. The ith *component* is the embedding of the ith copy of S^1 into S^3.

All of the definitions above can be generalized to links. Write down definitions for the following:

(1) oriented link $L = (L, o)$ If we want to be explicit about the orientation of a link, we use the letter o for the orientation information.
(2) unoriented (isotopy class of) L
(3) $m(L)$, the mirror of a link L
(4) L^r, the reverse of an oriented link (L, o)

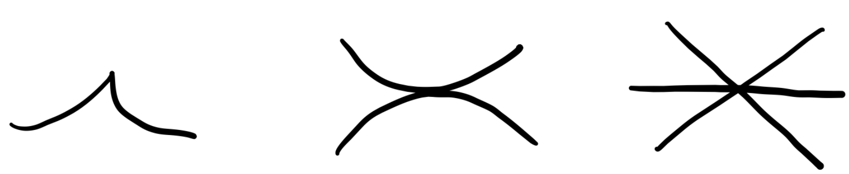
How many orientations does a link with ℓ components have?

Remark 2.1.10. There are also other categories of links, such as *topological links* and *wild links*. We won't talk about these until, maybe, far later in the course.

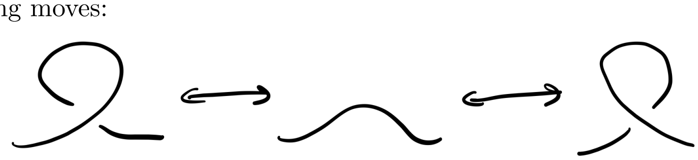
2.2. **Link diagrams and Reidemeister moves.** We will now start abusing notation and language without comment, per Remark 2.1.3.

We've so far been implicitly using link diagrams to draw links. More formally, a *link diagram* is a compact projection of a representative of a link L onto the xy-plane such that the only intersections are transverse double points.

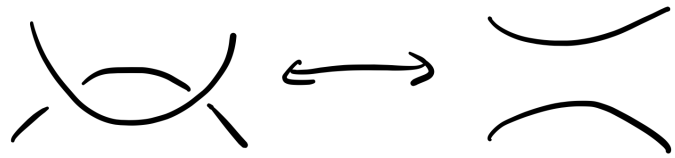
For example, here are some bad singularities:

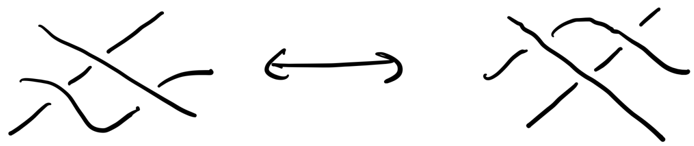


Theorem 2.2.1 (Reidemeister, (add: cite), 1930s)**.** If D and D' are two diagrams of the same link, then they are related by a finite sequence of the following moves:



(R1)

(R2) 

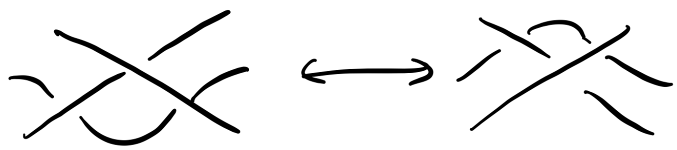
(R3) 

Remark 2.2.2. (1) These are local pictures. You can rotate them.

(2) Note that there are technically many cases within each 'move', if you consider all the possible orientations of the interacting strands in these local pictures. This is important when producing invariants for oriented links.

(3) If D and D' are diagrams for the same link, then we will sometimes write $D \sim D'$. In other words, "there exists a sequence of Reidemeister moves between" is an equivalence relations on the set of link diagrams.

Exercise 2.2.3. (Unimportant) Explain why we don't need to include the following local move:



Definition 2.2.4. A *link invariant* F valued in a category $\mathcal{C}$ is a machine that

- takes in a link diagram D
- and outputs $F(D) \in \mathrm{Ob}(\mathcal{C})$

such that

$$D \sim D' \qquad \Longrightarrow \qquad F(D) = F(D').$$

Remark 2.2.5. Note that a link invariant is not necessarily a functor. We will define the categories **Link** and **LinkDiag**, and define *functorial* link invariants later.

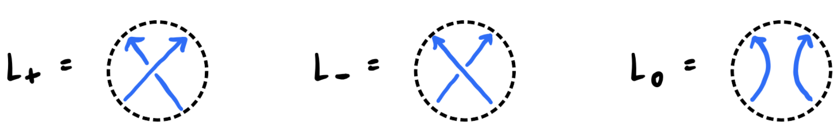
2.3. **Jones polynomial via skein relation.** Khovanov homology is a categorical lift of the Jones polynomial, so we will focus on this invariant first.

Remark 2.3.1. The first polynomial invariant of links was the Alexander polynomial, introduced in the 1920s. We will not discuss the Alexander polynomial until needed later.

The Jones polynomial was discovered by Vaughan Jones in the 1980s, and arose from his work in statistical mechanics. We will not study the original definition of his invariant, but you are welcome to look into it if it interests you. Perhaps a final project idea? (add: citation)

Definition 2.3.2. **(/ Theorem / Algorithm / Conventions)** The Jones polynomial is a link invariant valued in Laurent polynomials $\mathbb{Z}[q, q^{-1}]$ that is uniquely determined by the following recursion:

- base case: $J(\bigcirc) = 1$
- skein relation: $q^{-2}J(L_+) - q^2 J(L_-) = (q^{-1} - q)J(L_0)$ where



Remark 2.3.3. The diagrams for L_+, L_-, and L_0 above show a *positive crossing*, a *negative crossing*, and an *oriented resolution*, respectively. The singularity at the crossing can be resolved in two ways; L_0 is the only resolution that maintains the orientation of all the strands.

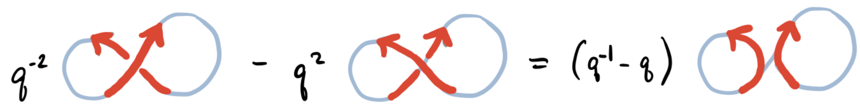
Remark 2.3.4. In class, we essentially used Jones' conventions; these produce a Laurent polynomial in the variable $\sqrt{t}$:

- base case: $V(\bigcirc) = 1$
- skein relation: $t^{-1}V(L_+) - tV(L_-) = (t^{1/2} - t^{-1/2})V(L_0)$.

If you look at a database of Jones polynomials, you'll likely find this convention used.

For the purposes of this course, we will stick with Khovanov's original conventions for the first few weeks of class.

Example 2.3.5. Here we use the skein relation to compute the Jones polynomial (with the conventions set in Definition 2.3.2) of the *unlink of two components*, $\bigcirc\bigcirc$.



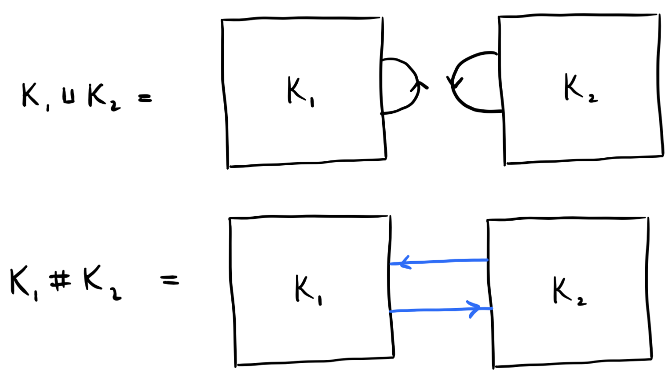
$$q^{-2} \cdot 1 - q^2 \cdot 1 = (q^{-1} - q)J(\bigcirc\bigcirc)$$
$$J(\bigcirc\bigcirc) = \frac{q^2 - q^{-2}}{q - q^{-1}} = q + q^{-1}.$$

Exercise 2.3.6. Compute the Jones polynomials of the following links:

(1) (positively linked) Hopf link
(2) right-handed trefoil

Exercise 2.3.7. (Important)

(1) Prove that for any link L, $J(L) = J(L^r)$.
(2) Let K_1 and K_2 be knots, and let $K_1\#K_2$ denote their *connected sum*[3]:



Prove that $J(K_1\#K_2) = J(K_1)J(K_2)$. What is $J(K_1 \sqcup K_2)$?

Exercise 2.3.8. (a) Show why the Jones polynomial is invariant under an R1 move.
(b) Prove that for any link L, $J(\bigcirc L) = (q + q^{-1})J(L)$.

Question 2.3.9. (Open) Does the Jones polynomial *detect* the unknot U? In other words, if $J(D) = 1$, is D necessarily a diagram for the unknot?

[3]I have not provided a precise definition here. This is a good time to Google the term yourself to see some examples of the connected sum operation.

2.4. **Jones polynomial from Kauffman bracket.** In this section, we generally follow [BN02], certainly with the same conventions. However, I may word things differently or rotate some pictures, for our future benefit.

Definition 2.4.1. The *Kauffman bracket* $\langle \cdot \rangle$ is defined by the recursion

- $\langle \varnothing \rangle = 1$
- $\langle \bigcirc L \rangle = (q + q^{-1}) \langle L \rangle$
- $\langle \times \rangle = \langle)(\rangle - q \langle \asymp \rangle$

Note that these local pictures are *unoriented*, unlike those in the skein relation for the Jones polynomial.

Remark 2.4.2. Each crossing has two possible smoothings, which we will name the 0-resolution and the 1-resolution, as shown:

$$)(\xleftarrow{0} \times \xrightarrow{1} \asymp.$$

- The 0-resolution is the one that you would naturally draw if you started at an over-strand and drew a smile. It gets no coefficient in the Kauffman bracket.
- The 1-resolution is the one that you would draw if you started at the over-strand and drew a frown. It gets a coefficient of $-q$ in the Kauffman bracket.

Note that I draw my smileys like this: 

The Kauffman bracket is *not* a link invariant. To see this, compare $\langle \bigcirc \rangle$ and $\langle \infty \rangle$. It is, however, a *framed* invariant. May talk more about this later.

To fix this, we have to take into account the *writhe* of a diagram, which measures how 'twisty' the choice of diagram is. 

Definition 2.4.3. Let D be an oriented link diagram.

- Let n be the number of crossings in D.
- Let n_+ be the number of positive crossings, and let n_- be the number of negative crossings.

The *writhe* of D is $\mathrm{wr}(D) = n_+ - n_-$.

Definition 2.4.4. The *(unnormalized)* Jones polynomial is defined by

$$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$$

Note that we are treating L as both the link diagram and the link it represents. Equivalently, $\hat{J}(L) = (-q)^{-n_-} \cdot q^{\mathrm{wr}(L)} \langle L \rangle$.

Remark 2.4.5. Alternatively, we can build the overall shifts into the bracket, using the following recursion:

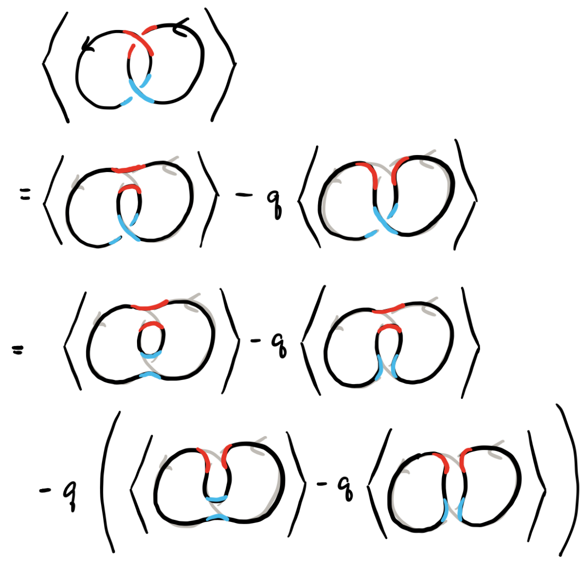
- $\langle \varnothing \rangle_o = 1$
- $\langle \bigcirc L \rangle_o = (q + q^{-1}) \langle L \rangle_o$ Recall that the two orientations on the unknot are isotopic.
- (positive crossing) $\langle \nwarrow\!\!\nearrow \rangle_o = q \langle)(\rangle_o - q^2 \langle \asymp \rangle_o$
- (negative crossing) $\langle \nwarrow\!\!\nearrow \rangle_o = q^{-2} \langle \asymp \rangle_o - q^{-1} \langle)(\rangle_o$

The notation '$\langle \cdot \rangle_o$' is not standard, and is used here only to distinguish it from the bracket used in [BN02].

Exercise 2.4.6. (1) How does the writhe of a diagram change under the Reidemeister moves?
(2) How does the Kauffman bracket $\langle \cdot \rangle$ of a diagram change under the Reidmeister moves?
(3) Verify that the bracket $\langle \cdot \rangle_o$ in Remark 2.4.5 computes $\hat{J}$, i.e. $\hat{J}(L) = \langle L \rangle_o$.

Example 2.4.7. Let H denote (oriented) Hopf link shown below, where $n = 2$, $n_+ = 2$, and $n_- = 0$.

First, we compute the Kauffman bracket of the shown diagram:



$$
\begin{aligned}
&= (q+q^{-1})^2 - 2q(q+q^{-1}) + q^2(q+q^{-1})^2 \\
&= q^4 + q^2 + 1 + q^{-2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\hat{J}(H) &= (-1)^{n_-} q^{n_+ - 2n_-} \langle H \rangle \\
&= q^2(q^4+q^2+1+q^{-2}) = q^6+q^4+q^2+1.
\end{aligned}
$$

The normalized Jones polynomial, using these conventions, would be $\frac{\hat{J}(H)}{q+q^{-1}} = q^5 + q$.

2.5. **Jones polynomial via the Khovanov bracket.** Khovanov homology $\mathrm{Kh}(\cdot)$ is a $(\mathbb{Z} \oplus \mathbb{Z})$-graded (co)homology theory whose graded Euler characteristic recovers the Jones polynomial. The extra grading is usually called the *quantum grading* or the *internal grading*.[4]

Definition 2.5.1. Let $C = (\bigoplus_{i,j\in\mathbb{Z}} C^{i,j}, d)$ be a bigraded chain complex of $\mathbb{Z}$-modules where, for each j, $\bigoplus_{i\in\mathbb{Z}} C^{i,j}$ is finite rank.

The *graded Euler characteristic* of C is

$$
\chi_q(C) = \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \operatorname{rank} H^{i,j}(C).
$$

Recall that the Euler characteristic of a CW complex can be computed using any CW decomposition; in particular, you do not need to compute the differentials in the CW chain complex to determine the Euler characteristic.

Similarly, we do not need to know the differential d to compute $\chi_q(C)$. So, for now, we will define Bar-Natan's *Khovanov bracket*, which lift the recursion in Definition 2.4.1 to the level of bigraded chain complexes:

Definition 2.5.2. The *Khovanov bracket* is defined by the axioms

- $[\![\varnothing]\!] = (0 \to \underline{\mathbb{Z}} \to 0)$
- $[\![\bigcirc L]\!] = V \otimes [\![L]\!]$ where $V = \mathbb{Z}v_+ \oplus \mathbb{Z}v_-$ of graded dimension $q+q^{-1}$. We will discuss the actual definition later.
- $[\![\times]\!] = \mathrm{Tot}\left(0 \to \underline{[\![\asymp]\!]} \xrightarrow{d} [\![)(]\!]\{1\} \to 0\right)$, where $\{1\}$ means 'q-grading shift of 1' (see Notation 2.5.5).

The terms of each chain complex at homological grading 0 are underlined. The *totalization* functor flattens a multi-dimensional complex into a one-dimensional chain complex. Since we are not considering the differential right now, we will not carefully define this here. If the tensor product of chain complexes looks unfamiliar to you, this is something you should look up.

[4]Sometimes I may say 'degree' instead of grading. I will likely pontificate on the terms *grading* and *degree* later in the course.

Remark 2.5.3. (add: Maybe add a remark about Grothendieck groups.)

Theorem 2.5.4 (Khovanov, as interpreted by Bar-Natan)**.** Given a link diagram L with $n_\pm$ crossings of sign $\pm$ respectively, the associated Khovanov chain complex is given by

$$\mathrm{CKh}(L) = [\![L]\!][n_-]\{n_+ - 2n_-\}.$$

The square brackets indicate a shift in the homological grading (see Notation 2.5.5). Then

$$\hat{J}(L) = \chi_q(\mathrm{CKh}(L)) = \chi_q(\mathrm{Kh}(L)).$$

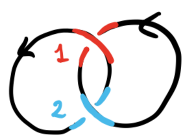
Notation 2.5.5. We set our conventions for the grading shift functors $[n]$ and $\{m\}$ to agree with those appearing in [BN02]. Let $A = \bigoplus A^{\bullet,\bullet}$ be a bigraded $\mathbb{Z}$-module. Then $A[n]\{m\}$ is the bigraded $\mathbb{Z}$-module where

$$(A[n]\{m\})^{i,j} = A^{i-n,j-m}.$$

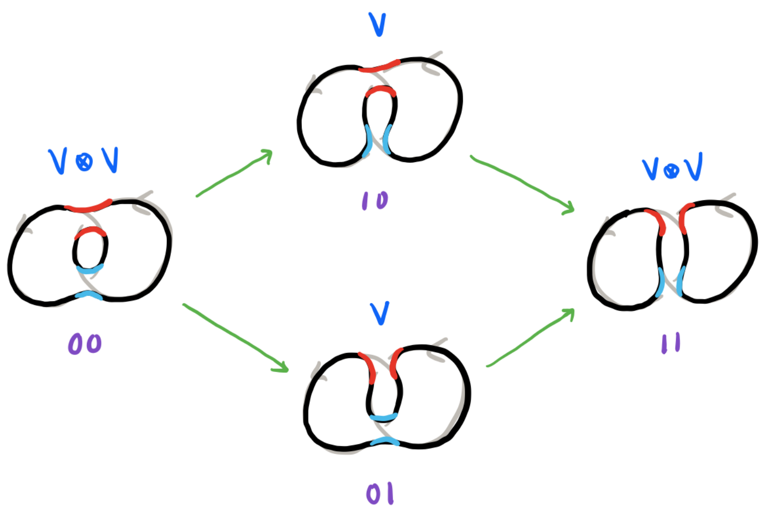
Visually, if A is plotted on the the $\mathbb{Z} \oplus \mathbb{Z}$ bigrading lattice, $A[n]\{m\}$ is obtained by grabbing A and moving it by the vector $ne_1 + me_2$.

Example 2.5.6. We now reorganize the computation in Example 2.4.7, as a primer for our formal introduction to Khovanov homology in the next section.

Let H denote both the following diagram as well as the underlying oriented Hopf link:



To compute $[\![H]\!]$, we draw the *cube of complete resolutions*:



We associate $V^{\otimes c}$ to each complete resolution containing c closed components. Each complete resolution corresponds to a binary string $u \in \{0,1\}^2$, and we perform a quantum grading shift of $|u|$ on associated $\mathbb{Z}$-module, where $|u|$ is the number of 1's appearing in the bitstring u.

This yields the Khovanov bracket

$$[\![H]\!] = (\underline{V \otimes V} \to V \oplus V\{1\} \to V \otimes V\{2\}).$$

(Recall that the underline indicates the chain group at homological grading 0.)

The Khovanov chain complex (without specifying differentials) is therefore

$$\begin{aligned}\mathrm{CKh}(H) &= (\underline{V \otimes V} \to V \oplus V\{1\} \to V \otimes V\{2\})\,[n_-]\{n_+ - 2n_-\}\\ &= (\underline{V \otimes V} \to V \oplus V\{1\} \to V \otimes V\{2\})\,[0]\{2\}\\ &= (\underline{V \otimes V}\{2\} \to V \oplus V\{3\} \to V \otimes V\{4\}).\end{aligned}$$

Schematically, can visualize the bigraded chain groups as follows, where each • represents a copy of $\mathbb{Z}$:

				χ:
$\mathrm{gr}_q = 6$			•	$0-0+1=1$
$\mathrm{gr}_q = 4$	•	••	••	$1-2+2=1$
$\mathrm{gr}_q = 2$	••	••	•	$2-2+1=1$
$\mathrm{gr}_q = 0$	•			$1-0+0=1$
	$\mathrm{gr}_h = 0$	$\mathrm{gr}_h = 1$	$\mathrm{gr}_h = 2$	

Here gr_h and gr_q denote the homological and quantum grading, respectively.

We conclude that, indeed, $\chi_q(\mathrm{CKh}(H)) = q^6 + q^4 + q^2 + 1 = \hat{J}(H)$.

Example 2.5.7. For a very comprehensive computation of the Jones polynomial of the trefoil using the Kauffman bracket, see Equation (1) of [BN02].

3. Khovanov homology

3.1. **The Khovanov chain complex** $\mathrm{CKh}(D)$**.** Given an oriented link diagram D representing a link L, the *Khovanov chain complex* $\mathrm{CKh}(D)$ is a $(\mathbb{Z} \oplus \mathbb{Z})$-graded chain complex of abelian groups. We define this chain complex throughout this section. Throughout, we will use the term *bigraded* in place of '$(\mathbb{Z} \oplus \mathbb{Z})$-graded'.

3.1.1. *Cube of resolutions.* The cube of resolutions will set up the topological story that the chain complex tells. Afterwards, we will replace each part of the cube of resolution with algebraic objects, to fully define the Khovanov chain complex.

Let n be the number of crossings in the diagram D, and pick an ordering for the crossings. Let c_i denote the ith crossing.

The n-dimensional *binary cube* $\{0,1\}^n$ is the poset of *binary strings* (a.k.a. *bitstrings*) of length n, with partial order given by $0 \prec 1$ and lexicographic order.

Notation 3.1.1. Recall some terms and notations used when discussing partially ordered sets:

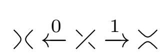
- If $u \prec v$, then we say u *precedes* v (and v *succeeds* u).
- If $u \prec v$ and they differ at exactly one bit, then we say u is an *immediate predecessor* of v (and v is an *immediate successor* of u). In this case, we write $u \prec_1 v$.

We may think of a poset as a directed graph, with and edge $u \to v$ if $u \prec_1 v$.

Notation 3.1.2. For convenience, if $u \prec_1 v$ and they differ at the ith bit, then we use the following length-n string in $\{0,1,*\}^n$ to denote the edge $u \to v$:

$$u_1 u_2 \cdots u_{i-1} * u_{i+1} \cdots u_n = v_1 v_2 \cdots v_{i-1} * v_{i+1} \cdots v_n$$

For $u \in \{0,1\}^n$, let D_u denote the complete resolution of the diagram D where crossing c_i is smoothed according to the bit u_i:



If $u \prec_1 v$, then D_u and D_v differ only in the neighborhood of one crossing, called the *active crossing* of the edge $u \prec_1 v$. All other crossings are *passive*.

Also, $u \prec_1 v$ represents either

- a *merge* of two *circles* (i.e. closed components) of D_u into one circle in D_v or
- a *split* of one circle of D_u into two in D_v.

Any of these circles are called *active circles* of the edge $u \prec_1 v$. All other circles are *passive*; they look the same in both D_u and D_v.

Finally, when assigning gradings, we will need the *Hamming weight* of bitstrings:

$$|u| = \sum_i u_i.$$

3.1.2. *Chain groups, distinguished generators, gradings. We now describe the* distinguished generators *of the Khovanov chain complex.*

Our chain groups will be bigraded by $\mathrm{gr} = (\mathrm{gr}_h, \mathrm{gr}_q)$:

- The first grading is called the *homological* grading, denoted gr_h. Its shift functor is denoted by $[\cdot]$.
- The second grading is called the *quantum* (or *internal*) grading, denoted gr_q. Its shift functor is denoted by $\{\cdot\}$.

Let V denote the bigraded $\mathbb{Z}$-module $\mathbb{Z}v_+ \oplus \mathbb{Z}v_-$ with generators $v_\pm$ in bigradings $(0, \pm 1)$.

Let $|D_u|$ denote the number of circles in the resolution D_u. The chain group lying above vertex u of the cube is $V^{\otimes |D_u|}[|u|]\{|u|\}$ generated by the 2^n length-n pure tensors

$$\{v_\pm \otimes \cdots \otimes v_\pm\}.$$

These are called the *distinguished* (i.e. chosen) generators of the chain group at this vertex.

Remark 3.1.3. The bigrading gr for a distinguished generator in $V^{\otimes k}$ is determined by the bigrading on V:

$$\mathrm{gr}(a \otimes v_\pm) = \mathrm{gr}(a) + \mathrm{gr}(v_\pm).$$

For example, $\mathrm{gr}(v_+ \otimes v_- \otimes v_-) = (0, -1)$.

Remark 3.1.4. Note that when we actually write down a chain complex, we implicitly homologically shift the chain groups. For example, the following chain complex is acyclic[5]:

$$\underline{V} \xrightarrow{1} V.$$

If we view the chain complex as a graded module C, and the differential as an endomorphism d, then

- $C = V \oplus V[1]$ and
- $d = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

This *mapping cone* point of view will become useful later in the course. Now is a good time to review/read about *mapping cones* in homological algebra.

3.1.3. *Differentials.* To each edge of the cube $u \to v$, we assign a map according to whether the edge represents a merging of two circles or the splitting of one circle. We casually call these the 'edge maps'.

- If $u \to v$ represents a merge, the map on tensor components corresponding to active circles is given by:

$$\begin{aligned} m : V \otimes V &\to V \\ v_+ \otimes v_+ &\mapsto v_+ \\ v_+ \otimes v_-, v_- \otimes v_+ &\mapsto v_- \\ v_- \otimes v_- &\mapsto 0 \end{aligned}$$

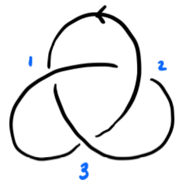
- If $u \to v$ represents a split, the map on active components is given by:

$$\begin{aligned} \Delta : V &\to V \otimes V \\ v_+ &\mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- &\mapsto v_- \otimes v_- \end{aligned}$$

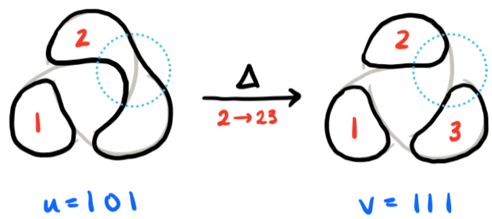
The map is identity on the passive components of the tensor product.

[5] i.e. homology vanishes

Example 3.1.5. Here is a diagram for the right-handed trefoil, with a choice of ordering on the three crossings (in blue):

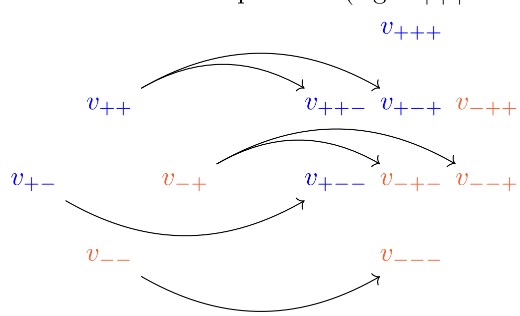


In the cube of resolutions, the edge corresponding to $101 \to 111$ is a merging of two circles into one, with the active crossing circled in dotted cyan:



We have chosen an ordering on the set of circles (in red) at each resolution so that we can identify the copies of V in the tensor product at each vertex. For example, the distinguished generator $v_- \otimes v_+$ at resolution D_u labels the smaller circle v_- and the larger v_+.

The linear map d_{uv} is given by the bundle of arrows shown below, where we use shorthand notation for compactness (e.g. $v_{+++} := v_+ \otimes v_+ \otimes v_+$):



Observe that d_{uv} is identity on the passive circle labeled '1' in both D_u and D_v.

Exercise 3.1.6. (Important)

(a) Verify that the merge and split maps, as written, decrease the quantum grading by 1.

So, as part of the differential, we modify the merge and split maps to be gr_q-preserving maps

$$d_{uv} : V \otimes V \to V\{1\} \qquad \text{or} \qquad d_{uv} : V \to V \otimes V\{1\},$$

depending on whether the edge $u \to v$ corresponds to a merge or a split, respectively.

(b) Verify that along each 2D face of the binary cube, the edge maps commute.

Therefore, in order to get a chain complex (where $d^2 = 0$), we will need to add some signs so that the faces instead *anticommute*.

(c) For an edge $u \to v$ in the cube with active crossing c_i, associate the following sign:

$$s_{uv} = (-1)^{\sum_{j=1}^{i-1} u_i}.$$

In other words, s_{uv} measures the parity of the number of 1's appearing before the $*$ in the label given to the edge in the binary cube (see Notation 3.1.2).

Verify that, for any face of the binary cube, and odd number of the four edges bounding that face will have sign assignment -1.

This is not the only possible sign assignment. (add: Discussion about cochains on cube.)

3.1.4. *Global grading shifts and homology.* To compute Khovanov homology of an oriented link L using diagram D:

(1) Draw the cube of resolutions.
(2) Associate modules $V^{\otimes |D_u|}$ to each vertex u of the cube.
(3) Associate linear maps $s_{uv} d_{uv}$ to each edge $u \to v$.
(4) Flatten the complex, by taking direct sums along Hamming weights; the resulting complex is the Khovanov bracket, $[\![D]\!]$.
(5) Add in the global bigrading shift $\{-n_-\}[n_+ - 2n_-]$ to get the *Khovanov chain complex*:

$$\mathrm{CKh}(D) = [\![D]\!]\{-n_-\}[n_+ - 2n_-].$$

(6) Take homology to get *Khovanov homology*:

$$\mathrm{Kh}(L) = H^*(\mathrm{CKh}(D)).$$

The Khovanov chain complex $\mathrm{CKh}(D)$ is bigraded, and its differential d_{Kh} is a bidegree $(1,0)$ endomorphism. Therefore $\mathrm{CKh}(D)$ is really the direct sum of many chain complexes, one for each quantum grading:

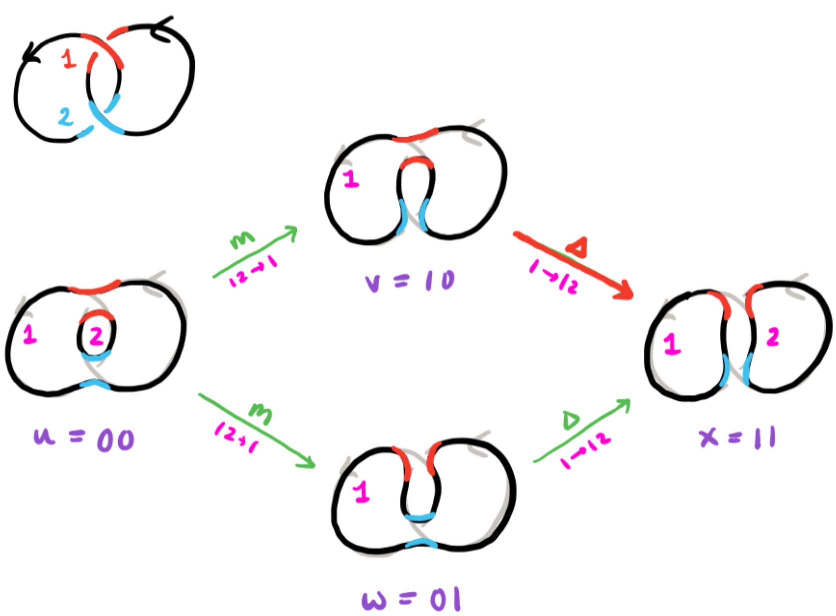
$$\mathrm{CKh}(D) = \bigoplus_{j \in \mathbb{Z}} \mathrm{CKh}^{\bullet, j}(D).$$

The homology is bigraded by the homological and quantum gradings (indexed below by i and j, respectively):

$$\mathrm{Kh}(L) = \bigoplus_{i,j \in \mathbb{Z}} \mathrm{Kh}^{i,j}(L).$$

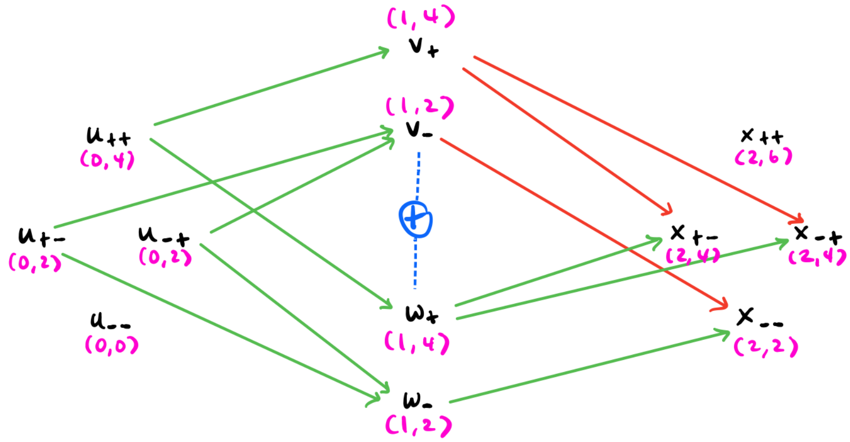
Example 3.1.7. We now finally compute the Khovanov homology of the Hopf link H from Examples 2.4.7 and 2.5.6.

In order to uniquely identify the generators at the different resolutions in the cube, we label the resolutions u, v, w, x and use these letters to denote the distinguished generators at each resolution:



- The vertices of the binary cube are labeled in purple.
- The ordering of the tensor factors (i.e. circles) at each resolution is shown in pink. Below each arrow, the pink text indicates the active circles in the source and target of that edge.
- The only edge with sign assignment -1 is the split map shown in bold red.

In shorthand, the Khovanov chain complex $\mathrm{CKh}(H)$ (i.e. with global shifts incorporated) is the following chain complex (direct sums are taken vertically):



The bigradings of the distinguished generators of $\mathrm{CKh}(H)$ are shown in pink.

The reader may now verify that the Khovanov homology of the positively-linked Hopf link H is the bigraded $\mathbb{Z}$-module

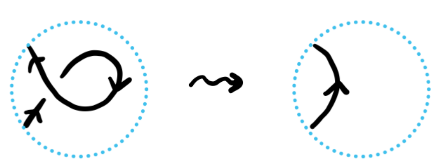
$\mathrm{gr}_q = 6$			$\mathbb{Z}$
$\mathrm{gr}_q = 4$			$\mathbb{Z}$
$\mathrm{gr}_q = 2$	$\mathbb{Z}$		
$\mathrm{gr}_q = 0$	$\mathbb{Z}$		
	$\mathrm{gr}_h = 0$	$\mathrm{gr}_h = 1$	$\mathrm{gr}_h = 2$

generated by the homology classes

$\mathrm{gr}_q = 6$			$[x_{++}]$
$\mathrm{gr}_q = 4$			$[x_{+-}] = [x_{-+}]$
$\mathrm{gr}_q = 2$	$[u_{+-} - u_{-+}]$		
$\mathrm{gr}_q = 0$	$[u_{--}]$		
	$\mathrm{gr}_h = 0$	$\mathrm{gr}_h = 1$	$\mathrm{gr}_h = 2$

Exercise 3.1.8. The diagrams D and D' below both represent the unknot, U.

(a) Compute the Khovanov chain complex CKh for both, and then compute homology to verify that they indeed agree.
(b) Can you prove that Khovanov homology is invariant under the following Reidemeister 1 move?



Bar-Natan's introductory paper does cover this, but try first to use intuition from your solution to part (a) to find the appropriate chain homotopy equivalence.

3.2. **An underlying TQFT.** The algorithm/definition in the previous section is concrete, but you might be wondering where these m and Δ maps come from. The answer lies in the fact that morphisms in the category of bigraded $\mathbb{Z}$-modules are images of cobordisms under a functor from a more topologically defined category. In this section, we describe an[6] underlying TQFT that determines the maps m and Δ from the previous section.

3.2.1. *TQFTs.* QFTs (quantum field theories) and TQFTs (topological quantum field theories), have a rich history in mathematical physics. The purposes of this course, we will use the following simplified (i.e. vague) definition, adapted from [Ati88]:

Definition 3.2.1. (Vague[7]) Let R be a commutative ground ring. An $(n+1)$-dimensional *TQFT* is a functor Z from a category of closed n-dimensional manifolds and $(n+1)$-dimensional cobordisms[8] between them to a category of finitely generated (see Remark 3.2.3) R-modules, such that

- Z is *multiplicative*: $Z(Y \sqcup Y') = Z(Y) \otimes Z(Y')$.
- Z is *involutory*: If $\bar{Y}$ is Y with the opposite orientation, then $Z(\bar{Y}) = Z(Y)^*$ (the dual module).

We also naturally would like Z to send the identity cobordism $Y \times I : Y \to Y$ to the identity map.

Functoriality implies that if $C_{01} : Y_0 \to Y_1$ and $C_{12} : Y_1 \to Y_2$ are cobordisms, then

$$Z(C_{12} \circ C_{01}) = Z(C_{12}) \circ Z(C_{01}) : Z(Y_0) \to Z(Y_2).$$

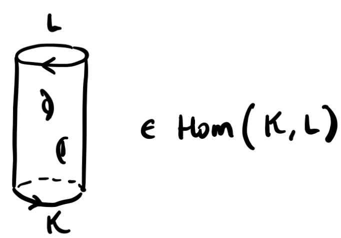
Make sure you know what *category* and *functor* mean. Things will get confusing from here on out if these terms aren't clear.

Remark 3.2.2. To learn more about physical origins of the term *topological quantum field theory*, take a look at the ncatlab page.

Atiyah's *Topology quantum field theories* [Ati88] provides a set of precise axioms for $(n+1)$-TQFTs and surveys some prominent examples.

Remark 3.2.3. Since any (orientable) manifold only has two orientations, the category of cobordisms is *pivotal*, i.e. $A \cong (A^*)^*$ for any object A.

(1) Thus the target category of the functor needs to be pivotal as well; this is why we require *finitely generated* R-modules.
(2) Since our categories are pivotal we can identify any cobordism $C : K \to L$ with various 'bent' versions of C, shown in the schematics below: (add: discussion of boundary orientation?)



[6]We will see variations later, but focus first on the one that's easiest to work with.
[7]We are more interested in specific TQFTs, so will not emphasize the details here.
[8]up to some equivalence relation

We thus get equivalences

$$\mathrm{Hom}(K, L) \cong \mathrm{Hom}(K \otimes \bar{L}, \varnothing) \cong \mathrm{Hom}(\varnothing, \bar{K} \otimes L).$$

(3) We can view the equivalence

$$\mathrm{Hom}(Z(K) \otimes Z(\bar{L}), Z(\varnothing)) \cong \mathrm{Hom}(Z(K), Z(L))$$

as an instance of the Tensor-Hom Adjunction:

$$\mathrm{Hom}(M \otimes N, P) \cong \mathrm{Hom}(M, \mathrm{Hom}(N, P))$$

by setting $M = Z(K)$, $N = Z(\bar{L}) = Z(L)^*$, and $P = Z(\varnothing) = R$. (Note that $\mathrm{Hom}(N, P) = \mathrm{Hom}(Z(L)^*, R) = (Z(L)^*)^* \cong Z(L)$.)

3.2.2. *Bar-Natan's dotted cobordism category and TQFT.* In this section, we will focus on one particular TQFT, which we will denote $\mathcal{F}_{\mathrm{BN}}$. This material is adapted from [BN05].

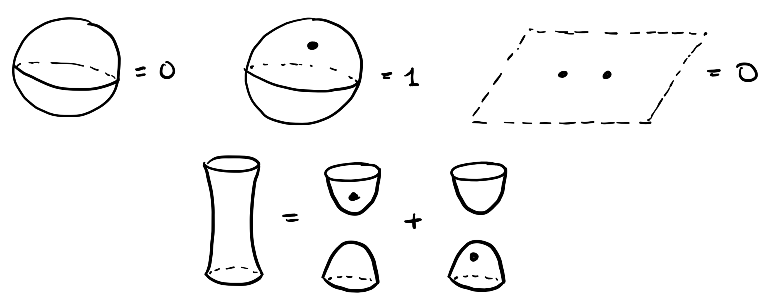
First, we need to understand the source category.

Definition 3.2.4. A small category $\mathcal{C}$ is *preadditive* (a.k.a. *$Mod_{\mathbb{Z}}$-enriched*)if for $X, Y \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group (i.e. a $\mathbb{Z}$-module) under composition, and this composition is bilinear (under the action of $\mathbb{Z}$).

Remark 3.2.5. A category is *additive* if additionally, we have finite coproducts. We will boost up to additive categories later in §3.4.

Definition 3.2.6. The preadditive category $\mathcal{TL}_0$ is defined as follows:

- Objects are closed 1-manifolds with finitely many components embedded smoothly in the plane $\mathbb{R}^2$; we call these *planar circles*. Notice that we are *not* identifying isotopic embeddings, but we *do* ignore parametrization.
- Morphisms are finite sums of *dotted cobordisms*, or smooth surfaces embedded in $\mathbb{R}^2 \times I$,
  - with boundary only in $\mathbb{R}^2 \times \partial I$,
  - possibly decorated with a finite number of dots,
  - up to boundary-preserving isotopy, and
  - subject to the following relations:



(1) ‘

The last relation is often referred to (gruesomely) as the *neck-cutting* relation.

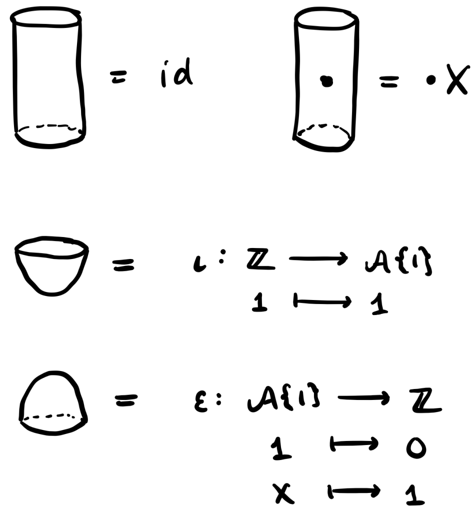
The target category will be bigraded $\mathbb{Z}$-modules, which we will denote as $gg\mathrm{Mod}_{\mathbb{Z}}$ when we want to emphasize the bigrading, or just $\mathrm{Mod}_{\mathbb{Z}}$ for short. The most important module for us will be the one associated to a single circle in the plane, which we previously called V. We will now rename this module and view it as $V \cong \mathcal{A}\{1\}$, where

$$\mathcal{A} = \mathbb{Z}[X]/(X^2)$$

with $\deg_q(X) = -2$.

Definition 3.2.7. The functor $\mathcal{F}_{\mathrm{BN}} : \mathcal{TL}_0 \to gg\mathrm{Mod}_{\mathbb{Z}}$ assigns

- to each collection of k planar circles the module $(\mathcal{A}\{1\})^{\otimes k}$ (with $\varnothing \rightsquigarrow \mathbb{Z}$)
- to each dotted cobordism a linear map determined by the following assignments:



Remark 3.2.8. Notice that Definition 3.2.7 is really a Definition-Theorem, since one needs to check that it really is a functor. What does one need to check to make sure that $\mathcal{F}_{\text{BN}}$ really is a functor?

Let C_1 and C_2 be cobordisms $\varnothing \to P$, where P is a collection of planar circles. Then there is a $\mathbb{Z}$-valued pairing

$$\langle C_1, C_2 \rangle = \mathcal{F}_{\text{BN}}(C_1 \cup_\partial \bar{C}_2) \qquad (= \mathcal{F}_{\text{BN}}(\bar{C}_1 \cup_\partial C_2))$$

because $C_1 \cup_\partial \bar{C}_2$ is a closed 2-manifold, and the relations in $\mathcal{TL}_0$ allow us to evaluate closed surfaces.

In the following exercise, we will use this pairing to recover the linear maps m and Δ from the previous section.

Exercise 3.2.9. *(Not eligible for HW submission; we will complete these in lecture.)*

(a) In Definition 3.2.7, why did we not need to specify how $\mathcal{F}_{\text{BN}}$ assigns higher-genus cobordisms?
(b) We can identify the distinguished $\mathbb{Z}$-module generators 1 and X in $\mathcal{A}\{1\}$ with a cup and a dotted cup, respectively. Why does this make sense?
(c) What is the dual cobordism to the cup? ...the dotted cup?
(d) Using this basis for $\mathcal{A}\{1\}$ and the pure tensor basis for $\mathcal{A}\{1\}^{\otimes 2}$, determine the matrix (i.e. chart) associated to the merge and split cobordisms by using the pairing.

Aside 3.2.10. (Computing linear maps using pairings)

Here is the linear algebra analogue to the method we are using above to compute the maps m and Δ.

Let $M : \mathbb{R}^n \to \mathbb{R}^n$ be a matrix with respect to the standard basis vectors $\{e_i\}$. We can view M as a pairing, by setting

$$\langle v, w \rangle_M = w^\top M v \in \mathbb{R}.$$

The entries of the matrix are determined by the pairing on basis vectors:

$$M_{ij} = e_i^\top M e_j$$

(Recall i is the row and j is the column.) This is because M_{ij} is the coefficient of e_i in the image vector Me_j.

In our setting, we are actually taking one additional step, which is to identify the column vectors v and w with the linear maps $\mathbb{R} \to \mathbb{R}^n$ that they represent, as $n \times 1$ matrices. (The 'dual' vector $w^\top$ is a row vector that represents a linear map $\mathbb{R}^n \to \mathbb{R}$.)

So a 'closed surface' in our setting corresponds to the composition of maps

$$\mathbb{R} \xrightarrow{v} \mathbb{R}^n \xrightarrow{M} \mathbb{R}^n \xrightarrow{w^\top} \mathbb{R},$$

and setting v and w to be basis vectors allows us to compute the entries of M.

(Note that any linear map $\mathbb{R} \to \mathbb{R}$ is necessarily of the form $\cdot r$ for some $r \in \mathbb{R}$. We are implicitly using $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, \mathbb{R}) \cong \mathbb{R}$.)

Observe that even though we defined the target of $\mathcal{F}_{\mathrm{BN}}$ to be $gg\mathrm{Mod}_{\mathbb{Z}}$, everything is happening at homological grading 0 at the moment; we don't yet have chain complexes! However, the quantum degree of morphisms can be determined in the source category $\mathcal{TL}_0$, as you'll discover in the next exercise.

Using Morse theory, we can show that every cobordism C between finite collections of planar circles $P \to P'$ can be isotoped so that the critical points occur at distinct times $t \in I$. We can slice up the cobordism into pieces

$$C = C_m \circ C_{m-1} \circ \cdots \circ C_2 \circ C_1$$

where each C_i is a disjoint union of some identity cylinders and one of the following four *elementary cobordisms*:

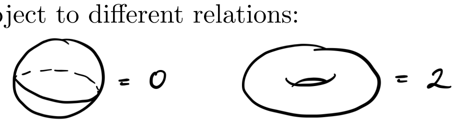
- cup (ι)
- cap (ε)
- merge (m)
- split (Δ)

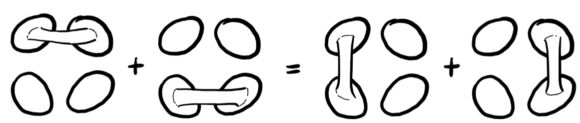
Exercise 3.2.11. (Important) Prove that for any cobordism $C : P \to P'$, the bidegree of the associated linear map is

$$\mathrm{gr}(\mathcal{F}_{\mathrm{BN}}(C)) = (0, \chi(C)),$$

where $\chi(C)$ is the Euler characteristic of the surface C.

Remark 3.2.12. Bar-Natan's more general cobordism category does not include dots as decorations. The objects are the same as in $\mathcal{TL}_0$, but morphisms are subject to different relations:

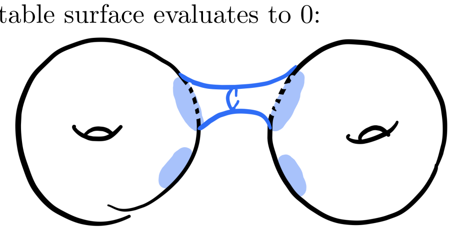




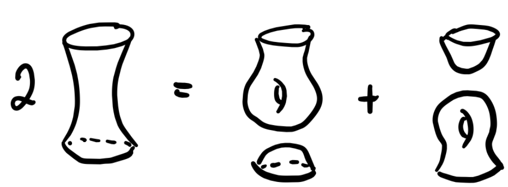
These relations are called the S (sphere), T (torus), and $4Tu$ (four tubes) relations.

We will sometimes work with this category, but for now have chosen to start with the dotted category because elementary calculations are easier there.

Exercise 3.2.13. (a) Use the T and $4Tu$ relations to show that the genus-2 orientable surface evaluates to 0:



(b) Use $4Tu$ to show the relation below:



Use this to explain why, morally speaking, 'dot' = 'half of a handle'.

3.2.3. *(Aside) Frobenius algebras and (1+1)-TQFTs.* We now discuss why we switched from using the $\mathbb{Z}$-module

$$V = \mathbb{Z}v_+ \oplus \mathbb{Z}v_-$$

to the underlying $\mathbb{Z}$-module of the $\mathbb{Z}$-algebra

$$\mathcal{A} = \mathbb{Z}[X]/(X^2).$$

See [Kho06] for a reference.

Definition 3.2.14. A *Frobenius system* is the data $(R, A, \varepsilon, \Delta)$ consisting of

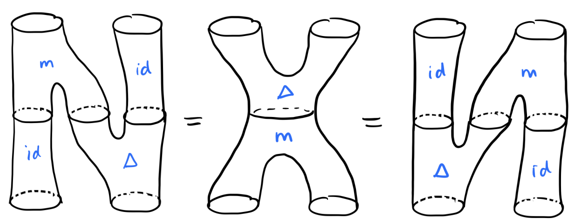
- a commutative ground ring R;
- an R-algebra A; in particular:
  - There is a *unit* or inclusion map $\iota : R \to A$ that sends $1 \mapsto 1$.
  - A has a *multiplication* map $m : A \otimes_R A \to A$.
- a *comultiplication map* $\Delta : A \to A \otimes_R A$ that is both coassociative and cocommutative; and
- an R-module *counit* map $\varepsilon : A \to R$ such that

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}.$$

The algebra A is a *Frobenius algebra*; it is simultaneously both an algebra and a coalgebra, and the following relation holds:

$$(\mathrm{id}_A \otimes m) \circ (\Delta \otimes \mathrm{id}_A) = \Delta \circ m = (m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta).$$

Remark 3.2.15. There are many equivalent definitions for the term 'Frobenius algebra'. The definition we used above is the most topological:



As you might have guessed, there is a strong relationship between Frobenius algebras and TQFTs.

In our case, $\mathcal{A} = \mathbb{Z}[X]/(X^2)$ is a rank 2 Frobenius extension of the ground ring $\mathbb{Z}$. Rank 2 Frobenius systems yield $(1+1)$-dimensional TQFTs, via the identifications below:

the $(1+1)$-TQFT sends this...	... to this in the Frobenius system
$\varnothing$	R
S^1	A
cup	ι
merge	m
split	Δ
cap	ε

By modifying the Frobenius algebra, we can get many more flavors of Khovanov homology. For example, if we instead use $R = \mathbb{Z}$, $A = \mathbb{Z}[X]/(X^2 - 1)$, we would build a version of Khovanov homology that is not quantum-graded (because $X^2 = 1$ is not a graded equation). This version of Khovanov homology is called *Lee homology*, and will be discussed later in this course when we talk about topological applications of Khovanov homology.

Warning 3.2.16. Do not use the category $\mathcal{TL}_0$ from Definition 3.2.6 as the source of the TQFT functor for Lee homology! The category $\mathcal{TL}_0$ was specifically tailored to the version of Khovanov homology over $\mathcal{A} = \mathbb{Z}[X]/(X^2)$, where 'two dots = 0'.

3.3. **Bar-Natan's tangle categories.** We now return back to dotted cobordisms, but develop the theory for not just planar circles, but also planar tangles.

Just as a link is an embedding of a finite number of circles in $\mathbb{R}^3$, a *tangle* is a proper embedding of a finite number of circles and arcs in a 3-ball B^3. A *planar tangle* is a tangle embedded in a 2-disk D^2.

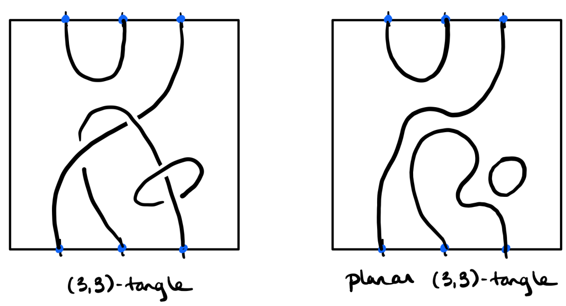
Since we will be building complicated categories out of tangles, we want to be very concrete with our definition of tangle categories, and will instead use the following definition.

Definition 3.3.1. An (n,n)*-tangle* is a 1-manifold with $2n$ boundary components (a.k.a. endpoints), properly embedded in the thickened square $[0,1]\times[0,1]\times(-\frac{1}{2},\frac{1}{2})$, with $2n$ endpoints located at

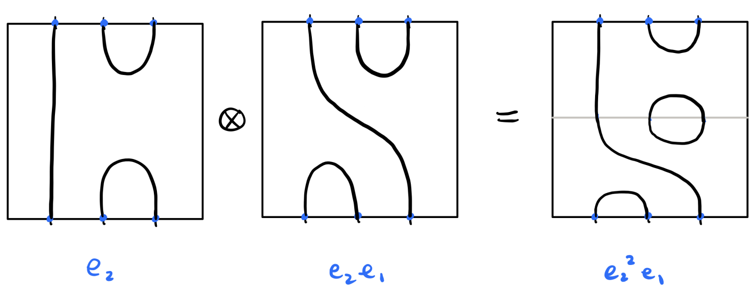
$$\left\{\left(\frac{i}{n+1},0,0\right)\right\}_{i=1}^{n} \cup \left\{\left(\frac{i}{n+1},1,0\right)\right\}_{i=1}^{n}. \tag{2}$$

An (n,n)*-tangle diagram* is a projection of an (n,n)-tangle to the unit square $[0,1]\times[0,1]$ where all singular points are transverse intersections (just as in link diagrams).

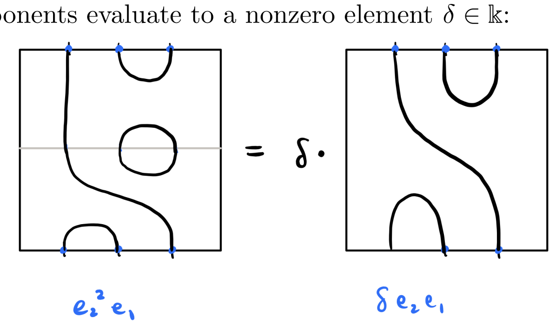
A *planar* (n,n)-tangle is an (n,n)-tangle embedded in the square $[0,1]\times[0,1]\times\{0\}$. In other words, a planar tangle is a crossingless projection of a (quite untangled) tangle.



Remark 3.3.2. A planar (n,n)-tangle with no closed components is a *crossingless matching.* These are the Catalan(n) many generators of the Temperley-Lieb algebra $\mathrm{TL}_n(\delta)$ over a field $\Bbbk$, whose composition $\otimes$ is given by stacking squares vertically:



Closed components evaluate to a nonzero element $\delta\in\Bbbk$:



We are not thinking about the monoidal structure of these diagrams just yet. But perhaps you can see the analogue between this (0+1)-dimensional cobordism category and the (1+1)-dimensional cobordism categories we're working with.

Definition 3.3.3 (cf. Definition 3.2.6)**.** The preadditive category $\mathcal{TL}_n$ is defined as follows:

- Objects are planar (n,n)-tangles with finitely many components, embedded smoothly in the square I^2 with endpoints at the $2n$ points specified in (2), denoted $\mathbf{p}$.
- Morphisms are finite sums of dotted cobordisms properly embedded in $I^2 \times [0,1]$,
  - with vertical boundary (i.e. on $\partial I^2 \times [0,1]$) consisting only of $2n$ vertical line segments $\mathbf{p} \times [0,1]$;
  - possibly decorated with a finite number of dots;
  - up to boundary-preserving isotopy;
  - subject to the same local relations (1) as in Definition 3.2.6.

So far, we've only defined a TQFT that can 'evaluate' closed components to a $\mathbb{Z}$-module. Since tangles in $\mathcal{TL}_n$ will inevitably have non-closed components, i.e. arcs, we will not be specifying a functor to $\mathrm{Mod}_{\mathbb{Z}}$ just yet. Instead, we will bring our algebraic tools to the topological categories $\mathcal{TL}_n$ and do as much homological algebra as we can before passing through any TQFT. Later on in the course, however, we may discuss how to define a 2-functor from the 2-category of (n points, (n,n)-tangles, (n,n)-tangle cobordisms) to an appropriately rich algebraic 2-category.

In particular, we can still treat $\mathcal{TL}_n$ as a (bi)graded category, where the quantum degree of a dotted cobordism C is *defined* as

$$\deg_q(C) := \chi(C) - n = \chi(C) - \frac{1}{2}(\#\text{ vertical boundary components})$$

(and the homological degree is $\deg_h(C) = 0$).

3.4. **Adding crossings : Boosting to chain complexes.** The Kauffman bracket showed us how to take a tangle with crossings and express it in terms of planar tangles; the Khovanov bracket tells us that a tangle with crossings is really just a chain complex of planar tangles. In this section, we will boost our categories $\mathcal{TL}_n$ to categories of chain complexes, so that we can capture tangles in general, not just planar tangles.

Throughout this section, we will be working in the more general setting of tangles; recall that $\mathcal{TL}_0$ is just a special case of the categories $\mathcal{TL}_n$. Also recall that these are all preadditive categories, by construction.

Definition 3.4.1 ([BN05], Definition 3.2)**.** Let $\mathcal{C}$ be a preadditive category. The *additive closure* of $\mathcal{C}$, denoted $\mathrm{Mat}(\mathcal{C})$, is the additive category defined as follows:

- Objects are (formal) direct sums of objects of $\mathcal{C}$. We can represent these as column vectors whose entries are objects of $\mathcal{C}$.
- Morphisms are matrices of morphisms in $\mathcal{C}$, which are added, multiplied, and applied to the objects just as matrices are added, multiplied, and applied to vectors.

An additive category is preadditive, by definition (Remark 3.2.5). We can build a category of chain complexes over any preadditive category:

Definition 3.4.2 ([BN05], Definition 3.3)**.** Let $\mathcal{C}$ be a preadditive category. The $\star$ *category of chain complexes* over $\mathcal{C}$ for $\star \in \{b, -, +\}$, denoted $\mathrm{Kom}^\star(\mathcal{C})$, is defined as follows:

- Objects are { finite length, bounded above, bounded below }[9] chain complexes of objects and morphisms in $\mathcal{C}$.
- Morphisms are chain maps between complexes.

Remark 3.4.3. For an example of a category of chain complexes over a preadditive but not additive category, take a look at $\mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$ in [ILM21, Remark 2.11].

Exercise 3.4.4. compute this composition of morphisms in Mat

[9]If the notation feels counterintuitive, just remember that 'bounded above' complexes are 'supported mostly in negative degrees'.

We are now ready to define the *Bar-Natan categories*, which are the categories we land in just before applying a TQFT to an algebraic category like chain complex over $\mathbb{Z}$-modules.

Definition 3.4.5. The Bar-Natan category $\mathcal{BN}_n^\star$ is $\text{Kom}^\star(\text{Mat}(\mathcal{TL}_n))$, for $\star \in \{b, -, +\}$. In practice, I will mostly drop the $\star$ from the notation. The bounded category is a full subcategory of both the $-$ and $+$ categories. We will only be working with bounded complexes at the beginning of the course, so the distinction won't matter.

These categories are rich enough to capture the entirety of the information in Khovanov homology, without ever passing to rings and modules. For example, we can think of Khovanov homology as a functor from links and cobordisms to the Bar-Natan category $\mathcal{BN}_0$. We may then choose a TQFT (or, equivalently, a Frobenius system) to apply to the resulting invariant, to obtain many different flavors of Khovanov homology. Moreover, we now have a "Khovanov homology for tangles."

Remark 3.4.6. Since $\mathscr{A} = \text{Mat}(\mathcal{TL}_n)$ is an abelian category, the *homotopy category* $K^\star(\mathscr{A})$ of chain complexes over $\mathscr{A}$ (for $\star \in \{b, -, +\}$) is a *triangulated category*. Khovanov homology can be thought of a functor to a triangulated category $K^b(\text{Mat}(\mathcal{TL}_0))$, in which case the quasi-isomorphism class of $\text{Kh}(L)$ is the link invariant. The Kauffman bracket skein relation gives exact triangles in this category. For exercises involving long exact sequences in Khovanov homology, see [Tur16].

To demonstrate the use of these Bar-Natan categories, we will use these categories to prove the invariance of Khovanov homology under some Reidemeister moves. You may find the proofs using the undotted (S, T, $4Tu$) theory in [BN05]. We will instead use the dotted theory as part of our demonstration. But first, we need to introduce two important lemmas that are immensely helpful with both by-hand and computer-assisted computations.

3.5. **Computational tools.** Bar-Natan's categories are also incredibly useful in computing Khovanov homology (by computer), because they allow for a 'divide-and-conquer' approach using tangles. Implemented algorithms typically scan a link diagram, simplifying the homological data at each step of the filtration of the diagram. The main tools used for simplification are *delooping* and *abstract Gaussian elimination*, which we discuss below. This section follows [BN07].

Remark 3.5.1. These algorithms have been immensely important to solving problems in low-dimensional topology. For example, Lisa Piccirillo's proof that the Conway knot is not slice [Pic20] involves computing the s-invariant (see §4.3) from the Khovanov homology of a knot with *a lot* (like around 40; I haven't counted carefully) of crossings, which is effectively impossible by naive computation.

Let us first write down a concrete formula for the degree of a morphism in $\mathcal{TL}_n$.

Warmup 3.5.2. As a sanity check, let's answer some warmup questions. See Notation 2.5.5 for our conventions on shift functors.

(a) Let A be a graded R-module, with shift functor $[\cdot]$. The set of degree-preserving maps $A[i] \to A[j]$ correspond to elements of $\text{Hom}^k(A, A)$ for some $k \in \mathbb{Z}$. What is k? Answer: $k = i - j$
(b) Suppose $\varphi \in \text{Hom}^0(A, A)$. What is the degree of the map induced by φ from $A[i] \to A[j]$? Answer: $j - i$
(c) Now suppose $\psi \in \text{Hom}^\ell(A, A)$. What is the degree of the map induced by ψ from $A[i] \to A[j]$? Answer: $j - i + \deg(\psi)$

Lemma 3.5.3. Let F be a (possibly dotted) cobordism between planar tangles $T, T' \in \mathcal{TL}_n$. The degree of the morphism

$$F : T\{i\} \to T'\{j\}$$

is

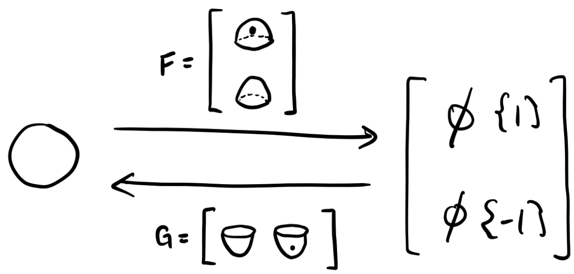
$$\deg_q(F) = j - i + \chi(F) - n \tag{3}$$

(where n is $\frac{1}{2}$ the number of tangle endpoints, or vertical boundary components).

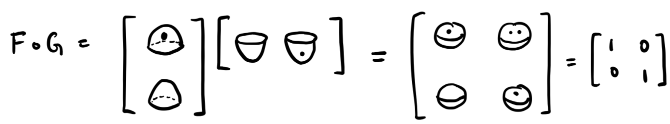
Verify that this makes sense to you!

From now on, we will treat diagrams and cobordism drawings as the same things as the objects and morphisms they represent in $\mathcal{BN}$.

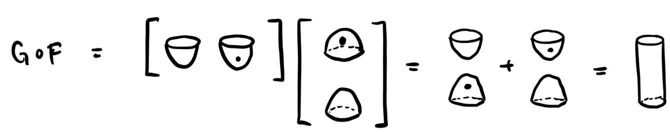
Lemma 3.5.4 ([BN07], Lemma 4.1)**.** (Delooping) $\bigcirc$ is chain homotopy equivalent to $\varnothing\{1\} \oplus \varnothing\{-1\}$ via the chain homotopy equivalences



Proof. We leave it to the reader to check that F and G are indeed degree-preserving (use Lemma 3.5.3). It suffices to check that $G \circ F \simeq \mathrm{id}_{\bigcirc}$ and $F \circ G \simeq \mathrm{id}_{\varnothing\{1\}\oplus\varnothing\{-1\}}$. Indeed,



by the sphere, dotted sphere, and two dots relations, and



by the neck-cutting relation. (No nontrivial homotopies were needed; these compositions are identity "on the nose".) ☐

Delooping essentially allows us to replace any flat tangle containing a closed component with two (quantum-shifted) copies of that flat tangle with that circle removed.

Our next tool is an abstract version of Gaussian elimination. You can ponder for yourself why this 'is' Gaussian elimination. Just remember that, in $\mathbb{Q}$, any nonzero number is a unit. Row operations are just changes of basis on the target of the linear map. Similarly, column operations are just changes of the basis on the source of the linear map.

Lemma 3.5.5 ([BN07], Lemma 4.2)**.** (Gaussian elimination) Let $\mathcal{C}$ be a pre-additive (add: check why BN said additive instead) category. Suppose in $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}))$ there is a chain complex segment

$$\cdots \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix}A\\ B\end{bmatrix} \xrightarrow{\begin{pmatrix}e & g\\ f & h\end{pmatrix}} \begin{bmatrix}C\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\gamma & \delta\end{pmatrix}} \cdots$$

where e is an isomorphism. Then the chain complex is homotopy equivalent to

$$\cdots \xrightarrow{\begin{pmatrix}\beta\end{pmatrix}} \begin{bmatrix}B\end{bmatrix} \xrightarrow{\begin{pmatrix}h - fe^{-1}g\end{pmatrix}} \begin{bmatrix}D\end{bmatrix} \xrightarrow{\begin{pmatrix}\delta\end{pmatrix}} \cdots$$

Note that

- A, B, C, D are objects in $\mathrm{Mat}(\mathcal{C})$
- $e, f, g, h, \alpha, \beta, \gamma, \delta$ are morphisms in $\mathrm{Mat}(\mathcal{C})$, i.e. these are matrices.

The main idea is that, by row and column operations, we are able to choose a basis so that the first chain complex is the direct sum of the second chain complex and an acyclic complex

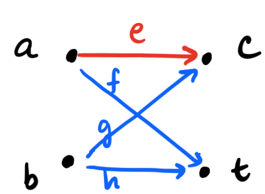
$$0 \to A \xrightarrow{e} C \to 0.$$

See Bar-Natan's proof for full details.

Corollary 3.5.6. (Cancellation lemma)[10] Suppose (C, d) is a chain complex freely generated by a distinguished set of generators $\mathcal{G}$, and we draw it using dots and arrows. For $x, y \in \mathcal{G}$, let $d(x, y)$ denote the coefficient of y in $d(x)$.

Suppose there is an isomorphism arrow $a \xrightarrow{e} c$ between distinguished basis elements $a, c \in \mathcal{G}$, i.e. the coefficient of the arrow is a unit in the ground ring R. Then (C, d) is chain homotopy equivalent to a 'smaller' chain complex (C', d') where C' is generated by $\mathcal{G} - \{a, c\}$, and for any $b \in \mathcal{G}$,

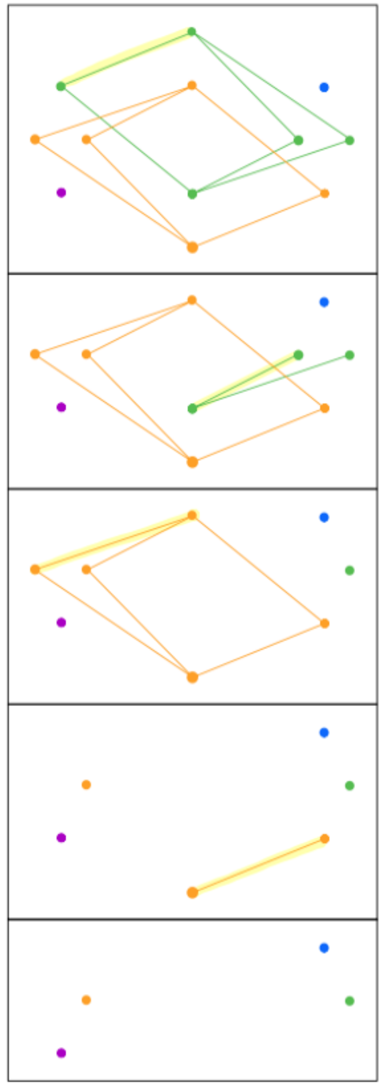
$$d'(b) = d(b) - d(b, c)d(a).$$



The new arrows ($fe^{-1}g$ in the figure above) are called *zigzags* for obvious reasons.

Remark 3.5.7. Cancellation provides an especially fast way to compute Khovanov homology over $\mathbb{F}_2$. Here is what my calculation for the Khovanov homology of the Hopf link looks like:

[10]See [BP10, Lemma 4.1], which directs you to [Ras03, Lemma 5.1], which directs you to [Flo89].



3.6. **Reidemeister invariance of Khovanov homology.** Bar-Natan's Reidemeister invariance proofs use the more general $S, T, 4Tu$ categories, and gives explicit homotopies where needed.

Here, we will stay with the $\mathcal{TL}_n$ categories that we defined and prove invariance under some (easy) Reidemeister moves, and make use of some facts from homological algebra.

Definition 3.6.1. Let $\mathcal{C}, \mathcal{C}'$ be chain complexes. Let C_n denote the n-th chain group of $\mathcal{C}$, and let C'_n be defined analogously.

- $\mathcal{C}'$ is a *subcomplex* of $\mathcal{C}$ (written $\mathcal{C}' \subseteq \mathcal{C}$) if each C'_n is a submodule of C_n and the differential on $\mathcal{C}'$ is the restriction of the differential on $\mathcal{C}$.
- If $\mathcal{C}' \subseteq \mathcal{C}$, then the *quotient complex* $\mathcal{C}/\mathcal{C}'$ has chain groups C_n/C'_n, and the differential is induced by the differential on $\mathcal{C}$.

Lemma 3.6.2. Let

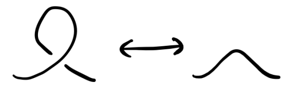
$$0 \to \mathcal{C}' \to \mathcal{C} \to \mathcal{C}'' \to 0$$

be an exact sequence of chain complexes, i.e. $\mathcal{C}'' \cong \mathcal{C}/\mathcal{C}'$.

(1) If $\mathcal{C}' \simeq 0$, then $\mathcal{C} \simeq \mathcal{C}''$.
(2) If $\mathcal{C}'' \simeq 0$, then $\mathcal{C} \simeq \mathcal{C}'$.

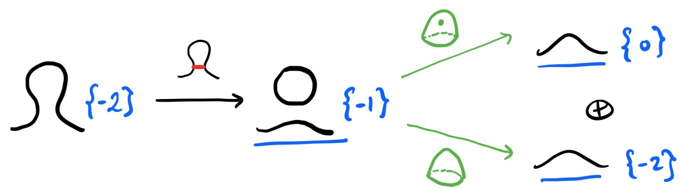
We now show R1 invariance of any flavor of Khovanov homology that factors through the dotted cobordism categories we defined.

Example 3.6.3. Consider the Reidemeister move involving a twist with a negative crossing:

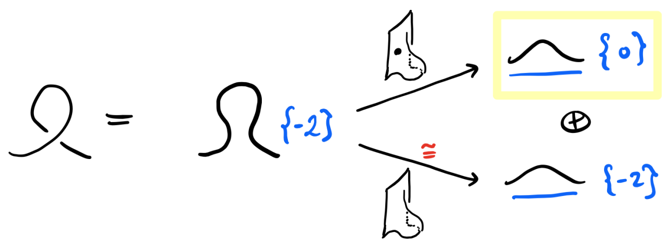


There is only one strand in this local picture, so this crossing will be negative regardless of how you orient the strand.

Using the Khovanov bracket, we resolve the crossing to obtain a two-term chain complex in $\mathrm{Kom}(\mathrm{Mat}(\mathcal{TL}_1))$ representing the twist, and deloop the resolution on the right:



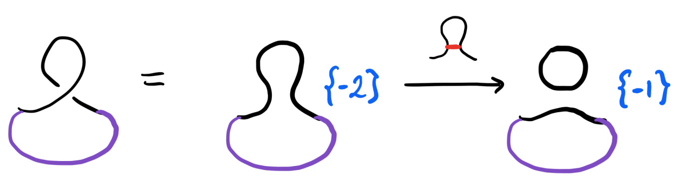
After delooping, our homotopic (actually isomorphic) complex is:

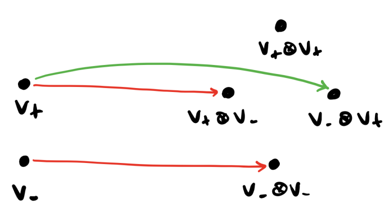


Notice that the downward arrow is an isomorphism in $\mathrm{Mat}(\mathcal{TL}_1)$. The quotient of the highlighted subcomplex is therefore nullhomotopic.

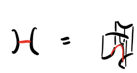
Using Lemma 3.6.2, we conclude that the complex representing the negative R1 twist is chain homotopy equivalent to the highlighted complex.

Remark 3.6.4. If you close up the $(1,1)$-tangle in the example above, then we can see exactly why the downward arrow is an isomorphism (red below), and the upward arrow (green below) is not:

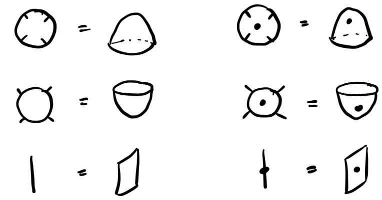




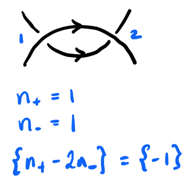
Notation 3.6.5. We have been using shorthand for merges and splits by drawing the descending manifold of the index-1 critical point in the saddle, on a diagram of the domain diagram:



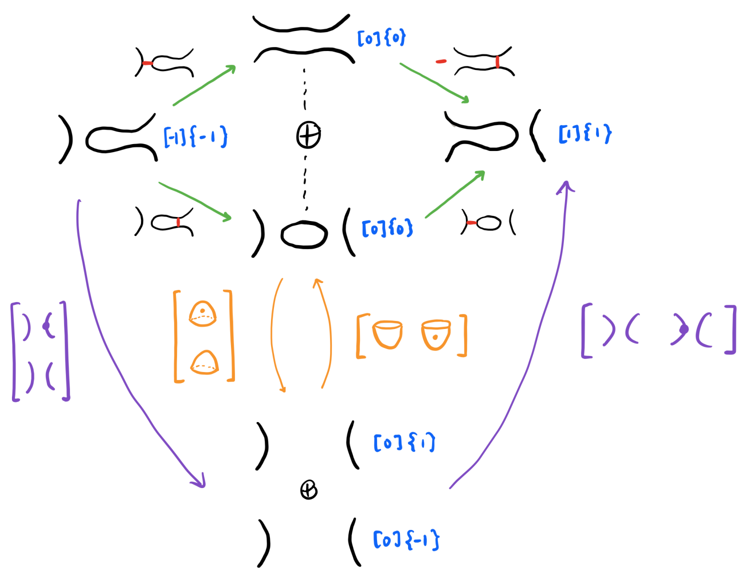
It is sometimes also useful to have shorthand for other types of elementary morphisms:



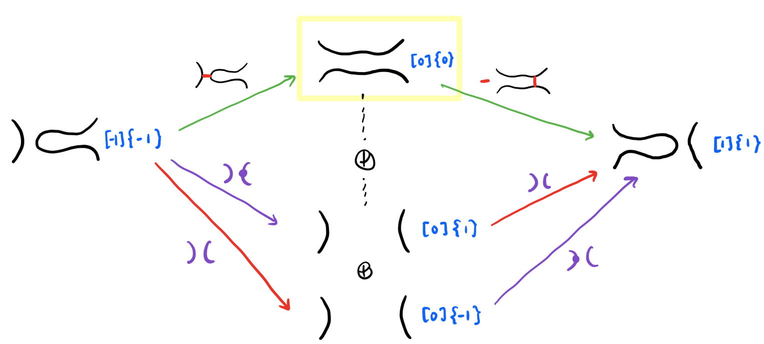
Example 3.6.6. Here is a proof of invariance under the 'braidlike' R2 move, corresponding to the diagram



Below is the complex representing the $(2,2)$-tangle above, with delooping maps draw in orange. The purple maps are compositions of green and orange maps.



After delooping, our complex looks like this:



Notice that the red arrows are isomorphisms. After performing Gaussian elimination on these red arrows, we obtain the homotopy equivalent complex highlighted in yellow.

R3 is always a more complicated move to deal with, because

- there are 3 crossings, so one needs to compare two cubes with eight vertices each, and
- there are three strands, and therefore many possible orientations

Nevertheless, Bar-Natan's proof fits beautifully on just one page; see [BN05, Figure 9] to learn what a 'monkey saddle' is.

Exercise 3.6.7. (1) Prove invariance under the other R1 move, where the twist has a positive crossing.
(2) Prove invariance under the other R2 move, where the strands are antiparallel.

3.7. **(Projective) Functoriality.** We are now equipped to fully define a functor that allows us to study links and cobordisms by replacing them with chain complexes and chain maps.

Once again, we begin by carefully defining our domain categories.

Definition 3.7.1. The category **Link** is defined as follows:

- Objects are smooth links in $\mathbb{R}^3$[11]
- Morphisms are cobordisms between links, modulo isotopy rel boundary.

At first glance, **Link** would be the category we would want to define our functor out of. However, recall that we don't actually compute Khovanov homology directly from links; we actually use link diagrams. So, we need to define an intermediate *diagrammatic category* that is equivalent to **Link**.

Definition 3.7.2. The category **LinkDiag** is defined as follows:

- Objects are smooth link diagrams drawn in $\mathbb{R}^2$ again, NOT up to isotopy!
- Morphisms are *movies* between link diagrams, modulo *movie moves*.

Movies and *movie moves* need to be discussed carefully, analogously to how we defined *link diagrams* and *Reidemeister moves*.

Definition 3.7.3. A *movie* is a finite composition of the following 'movie clips':

- planar isotopy
- Reidemeister moves
- Morse moves: birth of a circle, death of a circle, merging of two circles, splitting of one circle into two

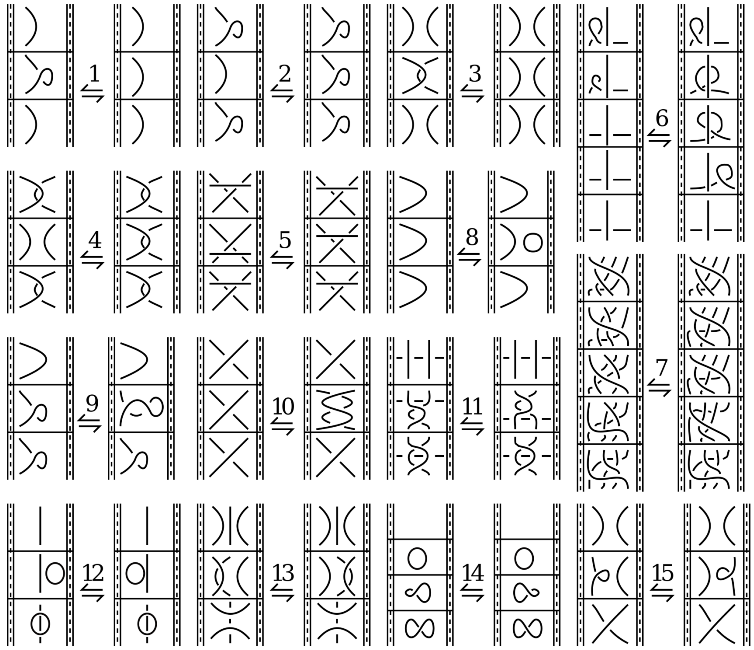
If $F \subset \mathbb{R}^3 \times I$ is a cobordism from $L_0 \subset \mathbb{R}^3 \times \{0\}$ to $L_1 \subset \mathbb{R}^3 \times \{1\}$, then an associated movie M can be thought of as a collection of movie 'frames' $\{M_t \mid t \in [0,1]\}$ where M_t is a diagrammatic projection of the 'slice' of the cobordism at time t, $F \cap (\mathbb{R}^3 \times \{t\})$. Indeed, the four Morse moves correspond to cup, cap, merge, and split cobordisms, respectively.

Just as we required link diagrams to only have transverse intersections, and for crossings to not be on top of each other, our definition for 'movie' ensures that our *critical frames* (i.e. frames where the projection $\mathbb{R}^3 \times \{t\} \to \mathbb{R}^2$ is not a link diagram) are isolated. Reidemeister moves and Morse moves are basically 'before and after' pictures of the process of passing through a critical frame.

In the category **Link**, morphisms are considered up to isotopy rel boundary. Just as Reidemeister proved that any diagram isotopy can be described as a finite composition of 3 local Reidemeister moves (and planar isotopy), Carter and Saito showed that any isotopy rel boundary of a cobordism can be captured as a finite composition of 15 local movie moves (and time-preserving isotopy), which are now known as the *Carter-Saito movie moves* [CS93].

Here is a figure ripped from [**?**]:

[11]or S^3, if preferred, but see Remark 3.7.7



Similar definitions can also be made for tangles:

Definition 3.7.4. The category $\mathbf{Tang}_n$ is defined as follows:

- Objects are smooth (n, n)-tangles in $[0, 1]^2$ with boundary at the $2n$ points $\mathbf{p}$ as in Definition 3.3.3.
- Morphisms are tangle cobordisms whose vertical boundary (i.e. the boundary in $\partial[0, 1]^2 \times I$) consists of the $2n$ line segments $\mathbf{p} \times I$, up to isotopy rel boundary.

The Carter-Saito movie moves are local, and so the definition of the diagrammatic category is essentially the same:

Definition 3.7.5. The category $\mathbf{TangDiag}_n$ is defined as follows:

- Objects are smooth (n, n)-tangle diagrams drawn in $[0, 1]^2$.
- Morphisms are *movies* between tangle diagrams that preserve the boundary points $\mathbf{p}$, modulo *movie moves*.

The punchline is that the diagrammatic categories are sufficient for capturing all the information in the topological categories.

Theorem 3.7.6. There is an equivalence of categories between **Link** and **LinkDiag** (and similarly between $\mathbf{Tang}_n$ and $\mathbf{TangDiag}_n$).

We can attribute this theorem to the combined work of Reidemeister and Carter–Saito but I should check this. The proof is beyond the scope of this course, but we will make use of these equivalences all the time.

Remark 3.7.7. If you want to work with links in S^3 instead, there are more diagrammatic moves to check. In particular, we need to include the *sweep-around* move, which equates the sweep-around movie (swinging a strand of the diagram around the 2-sphere through the point at infinity) with the identity movie. See [MWW22].

We now know what it means for a link invariant to be *functorial*: it is a functor from the category **Link** to its target category.

For our purposes, we will define the *Khovanov functor*

$$\mathcal{F}_{\mathrm{Kh}} : \mathbf{LinkDiag} \to gg\mathrm{Mod}_{\mathbb{Z}}$$

to be the composition of the functors

$$\mathbf{LinkDiag} \to \mathrm{Kom}(\mathrm{Mat}(\mathcal{TL}_0)) \to gg\mathrm{Mod}_{\mathbb{Z}}.$$

The first functor is the unfortunately unnamed (projective) functor that Bar-Natan gives us in [BN05][12] The second is the TQFT associated to the Frobenius system $(\mathbb{Z}, \mathcal{A} = \mathbb{Z}[X]/(X^2), \iota, m, \Delta, \varepsilon)$ where the morphisms are given by

$$\begin{aligned} \iota : \mathbb{Z} &\to \mathcal{A} \\ 1 &\mapsto 1 \end{aligned}$$

$$\begin{aligned} m : \mathcal{A} \otimes \mathcal{A} &\to \mathcal{A} \\ 1 \otimes 1 &\mapsto 1 \\ X \otimes 1, 1 \otimes X &\mapsto X \\ X \otimes X &\mapsto 0 \end{aligned}$$

$$\begin{aligned} \Delta : \mathcal{A} &\to \mathcal{A} \otimes \mathcal{A} \\ 1 &\mapsto X \otimes 1 + 1 \otimes X \\ X &\mapsto X \otimes X \end{aligned}$$

$$\begin{aligned} \varepsilon : \mathcal{A} &\to \mathbb{Z} \\ 1 &\mapsto 0 \\ X &\mapsto 1. \end{aligned}$$

Example 3.7.8. Khovanov homology over $\mathbb{F}_2$ is functorial; it defines a functor

$$\mathbf{LinkDiag} \xrightarrow{\mathcal{F}_{\mathrm{Kh}}} gg\mathrm{Vect}_{\mathbb{F}}$$

which, when precomposed with the equivalence

$$\mathbf{Link} \xrightarrow{\cong} \mathbf{LinkDiag},$$

defines a functor from the category of links to the category of bigraded vector spaces.

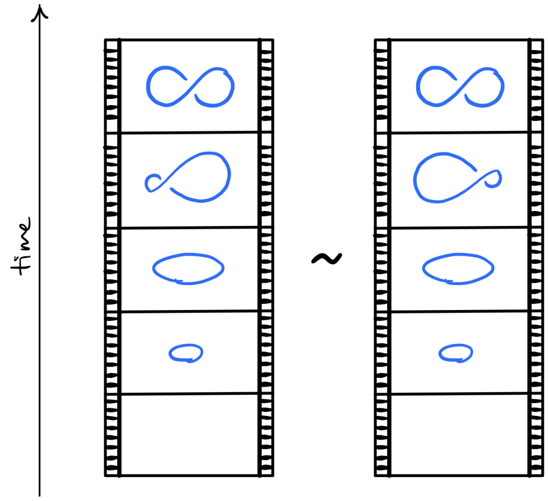
(add: Add MR citations) Jacobsson [?], Khovanov [?], and Bar-Natan [BN05, Section 4.3] each showed that Khovanov homology over $\mathbb{Z}$ is *projectively* functorial, meaning that it's functorial up to a sign. In other words, suppose you take two movies M and M' representing isotopic (rel boundary) cobordisms C and C'. Then the corresponding morphisms in $gg\mathrm{Mod}_{\mathbb{Z}}$ satisfy

$$\mathrm{Kh}(M') = \pm \mathrm{Kh}(M).$$

This sign discrepancy can be fixed, and it has been by a whole host of authors [Cap08, CMW09, Bla10, San21, Vog20, ETW18, BHPW23]. However, projective functoriality is enough for many, many important applications of Khovanov homology, which we will see in the next section.

Exercise 3.7.9. (Highly recommended) The following is a version of Movie Move 14 ('MM14'):

[12] We can't call it $\mathcal{F}_{BN}$ because 'Bar-Natan homology' means something else, which we will discuss later.



Compute the induced map on Khovanov homology for each movie, by using the Reidemeister 1 chain maps given in [BN05, Figure 5]. Show that the resulting morphisms have opposite signs.

You do not need to pass through the TQFT, but you are welcome to; the TQFT is a true (not projective) functor.

Beware: Bar-Natan's cobordisms flow *downward* with time. You can tell by looking at the domain and target objects of the cobordisms.

You can also quickly convince yourself that these movies, when read *backwards*, actually *do* yield morphisms with the same sign.

Maybe add section on general planar algebras, operads

4. Applications of Khovanov homology

As a functorial invariant for links in the 3-sphere and cobordisms in the 4-ball, Khovanov homology is a natural tool to study the relationship between links in the context of the surfaces they bound. In this section, we survey some of these applications. We start with some additional background on surfaces properly embedded in B^4. Once again, everything is smooth.

4.1. Surfaces in B^4.

We have already discussed cobordisms between links in S^3. In this section, we will use F to denote a cobordism (because they're 2-dimensional, like 'faces'). We reserve C for *concordances*, which are cobordisms that are diffeomorphic to cylinders. I.e. without the context of their embedding, they *are* cylinders. We will use the more precise term *annulus* instead of 'cylinder'.

Definition 4.1.1. Let K_0, K_1 be **knots** in S^3. A *concordance* C from K_0 to K_1 is an oriented cobordism such that $C \cong S^1 \times I$.

Equivalently, $C : K_0 \to K_1$ is a concordance if it is a (smooth, oriented) connected cobordism with $\chi(C) = 0$.

If such a C exists, then we say K_0 and K_1 are *concordant*: $K_0 \sim K_1$. This is an equivalence relation, and the equivalence classes are called *concordance classes*.

In fact, we can turn the set of knots into a group (!) by modding out by concordance:

Definition 4.1.2. The *smooth knot concordance group* $\mathscr{C}$ is the group where

- the elements are the concordance classes knots in S^3;
- the binary operation is induced by $\#$ (connected sum);
- the identity element is the class of *slice knots*, or knots that are concordant to the unknot;
- inverses are given by mirroring.

Remark 4.1.3. If you take a knife to a B^4 and cut off a slice, the cut you make is a slice disk. I haven't checked if this is the historical origin of the text.

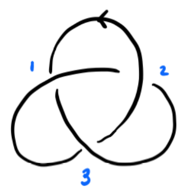
In general, a surface F properly embedded in B^4 whose boundary is $\partial F = K$ is called a *slice surface* for K.

Remark 4.1.4. We can equivalently say that a knot $K \subset S^3$ is *slice* if it bounds a disk D in B^4: if we arrange D so that it is in Morse position with respect to the radial function on B^4, then the boundary of a neighborhood of its lowest 0-handle is an unknot. In other words, an annulus is just a punctured disk.

In this case, we say that D is a *slice disk* for K.

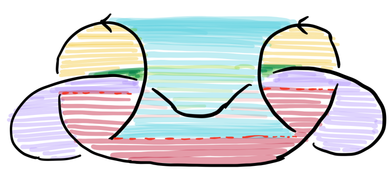
Again, this is actually a definition-theorem, and we will prove the theorem after seeing a quick example.

Example 4.1.5. Consider the right-handed trefoil K:

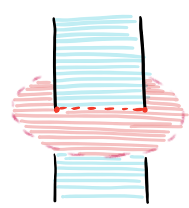


If we claim that the left-handed trefoil $m(K)$ represents the inverse concordance class, then we must show that $K\#m(K)$ is concordant to the unknot U.

It suffices to show that $K\#m(K)$ bounds a disk D embedded in B^4. You can see a projection of a slice disk in S^3 in the following picture:



The disk is immersed in S^3, and the only intersections are of the following form, called *ribbon intersections*:



The interior of the horizontal sheet can be pushed deeper into B^4, so that the slice disk in B^4 has no self-intersection.

To verify that the concordance group really is a group, we need to check that

(1) # is well-defined on equivalence classes
(2) # is associative
(3) the equivalence class $[U]$ acts as the identity
(4) for any K, $K\#m(K)$ is slice.

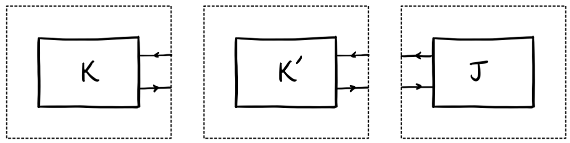
You can convince yourself that the binary operation # on the set of knots is both associative and commutative; if you want to think about this in more detail, see [Ada04]. Once we show that # is well-defined, showing that $[U]$ is the identity element is also easy.

It remains to show that # is well-defined, and that $K\#m(K)$ is slice. Both proofs use standard topological arguments.

Claim 4.1.6. # is a well-defined binary operation on the set of concordance classes.

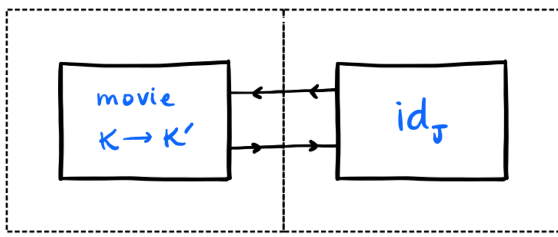
Proof. To see that # is a well-defined binary operation on concordance classes, consider knots $K, K', J \subset S^3$, where $K \sim K'$. Then there is some concordance $C : K \to K'$, a cobordism in $§^3 \times [0,1]$. Pick basepoints $p \in K$, $p' \in K'$ and isotope K' so that $p' = p$ (as points in S^3). Pick an arc $\gamma : [0,1] \hookrightarrow C$ such that $\gamma(0) = p$ and $\gamma(1) = p'$.

Perform a boundary-preserving ambient isotopy to 'straighten out' γ; that is, to arrange so that $\gamma(t) \in S^3 \times \{t\}$. (This is possible because the codimension of γ in $S^3 \times [0,1]$ is 3, and codimension 3 submanifolds can always be unknotted.) We will now assume that C is in such a position so that γ is of the form $p \times [0,1]$.



Now delete a small neighborhood of γ (i.e. $\nu(p) \times [0,1] \cap C$) in C. Pick a point $q \in J$ and delete it, forming a $(1,1)$ tangle $J - \nu(q)$ whose closure is J. Shrink this tangle so that it fits within $\nu(p)$.

Finally, glue $C - (\nu(p) \times [0,1])$ with $(J - \nu(q)) \times [0,1]$ to form a concordance from $K \# J$ to $K' \# J$.



□

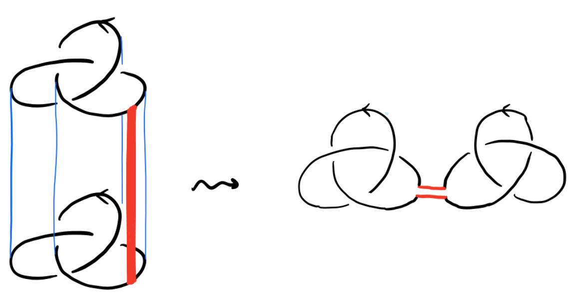
Aside 4.1.7. *How much should one write for such a proof?* You might notice that, for example, the last sentence of the preceding proof is not super duper precise. However, the figure helps you understand the notation, and also the underlying concept is quite simple. Also, we were careful to define the category **LinkDiag** so that we can make diagrammatic arguments.

As a human who uses language, I don't have a perfect answer for 'how much detail to show' – this comes from getting to know the common vocabulary, techniques, facts, and tricks used in the community you are writing for.

We also give a (more terse) proof sketch that $[m(K)] = [K]^{-1}$:

Claim 4.1.8. For any knot $K \subset S^3$, $K \# m(K)$ is slice.

Proof. The identity cobordism $K \to K$ is a concordance. Pick a point $p \in K$ and delete $p \times [0,1]$ from the identity cobordism. The resulting surface can be properly embedded in B^4 and viewed as a slice disk for $K \# m(K)$.



□

Remark 4.1.9. One can also define a notion of concordance between links. However, the notion of a 'link concordance group' isn't obvious, and is an area of active research.

4.2. **Obstruction to ribbon concordance.** In Example 4.1.5, we envisioned a slice disk by seeing that its projection to S^3 had only *ribbon singularities*. The following definition gives a more general definition of such phenomena.

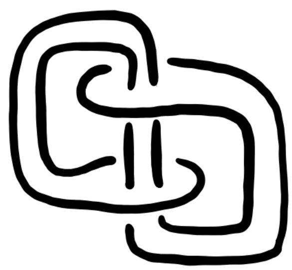
Definition 4.2.1. Let $F : L_0 \to L_1$ be a cobordism embedded in $S^3 \times I$ such that the second coordinate gives a Morse function. We say F is *ribbon* if, with respect to the Morse handle decomposition, F contains only 0- and 1-handles.

A ribbon cobordism that $D : \varnothing \to K$ that is a disk is called a *ribbon disk*, and is a special case of a slice disk.

A knot K that bounds a ribbon disk is called a *ribbon knot.*

From the definition, the 'ribbon concordant' relation is not symmetric. For example, the Stevedore knot 6_1 is slice: there is a clear ribbon concordance $C : U \to 6_1$. But if we reverse the Morse function, the upside-down concordance $\bar{C}$ clearly has a 2-handle.

Exercise 4.2.2. Draw a movie for the ribbon disk for 6_1 implied in the diagram below:



Remark 4.2.3. Unfortunately, we naturally would want to say '6_1 is concordant to the unknot', which is true, but not if we add the word 'ribbon'. Mathematically it makes more sense to say U is ribbon concordant to 6_1, and this is the language used in [LZ19]. Historically, some people defined ribbon surfaces by taking the descending Morse function and requiring only 1- and 2-handles, so be careful. For my sanity, I will always specify the direction of the cobordism as a morphism.

Remark 4.2.4. In terms of movies, a ribbon disk is a 'happy movie', where the only critical moments are births of circles and merges of circles. There are no scenes where circles split or die.

Conjecture 4.2.5. (Open: Slice-Ribbon Conjecture) All slice knots are ribbon.

It is clear that ribbon disks are slice, so any ribbon knot is a slice knot. However, not all slice *disks* are (isotopic to) ribbon disks (see Aru Ray's notes, Proposition 1.8). So, the conjecture posits that if K bounds a slice disk, it bounds a (potentially non-isotopic) ribbon disk.

Remark 4.2.6. We will see soon that there are knots that bound non-isotopic ribbon disks [HS21].

Levine–Zemke showed that we can use Khovanov homology to obstruct the existence of ribbon cobordisms between two knots:

Theorem 4.2.7 ([LZ19])**.** If $C : K_0 \to K_1$ is a ribbon concordance, then the morphism

$$\mathrm{Kh}(C) : \mathrm{Kh}(K_0) \to \mathrm{Kh}(L_1)$$

is injective, with left inverse $\mathrm{Kh}(\bar{C})$.

Here are some immediate corollaries that follow from basic algebra:

Corollary 4.2.8 ([LZ19])**.** Suppose $C : K_0 \to K_1$ is a ribbon concordance.

(1) At any bigrading, $\mathrm{Kh}^{i,j}(K_0) \hookrightarrow \mathrm{Kh}^{i,j}(K_1)$ as a direct summand.
(2) If additionally there is a ribbon concordance $C' : K_1 \to K_0$, then $\mathrm{Kh}(K_0) \cong \mathrm{Kh}(K_1)$.

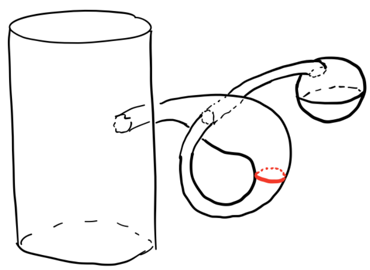
Seriously, prove these for yourself. There are four more corollaries in the paper, if you're interested.

The proof uses a topological lemma first appearing in Zemke's [Zem19]. This paper sparked a whole series of papers proving similar or related results for other functorial link homology theories (add: all citations).

The proof of this lemma is embedded in the proof of Zemke's main theorem:

Lemma 4.2.9 ([Zem19])**.** Let $C : K_0 \to K_1$ be a ribbon concordance, and let $\bar{C}$ be the upside-down (and opposite orientation) concordance to C, a morphism $K_1 \to K_0$.

If a movie presentation for C has n births and n saddles (note that the Euler characteristic of a concordance is 0), then there is a movie presentation for $\bar{C} \circ C$ with n births, n merge saddles, n split saddles that are dual to the merge saddles, and n deaths.



In particular, the lemma tells us that $\bar{C} \circ C$ is isotopic to a cobordism $K_0 \to K_0$ that looks like the identity cobordism for K_0 with n spheres tubed on. Because of nontrivial knot theory for surfaces in dimension 4, this cobordism might not be isotopic to the identity cobordism, but nevertheless, Khovanov homology can't tell because of the neckcutting relation:

Proof of Theorem 4.2.7. Take C and $\bar{C}$ as in Lemma 4.2.9. In $\mathcal{TL}_0$, by neckcutting, we see that the morphism $\bar{C} \circ C$ is equal to the morphism with 2^n summands that are all of the form id_{K_0} disjoint union with n spheres, and all summands have just one dot somewhere. After deleting the summands containing an undotted sphere, the only remaining morphism is the one consisting of id_{K_0} and n dotted spheres, which evaluate to a coefficient of 1. Therefore $\bar{C} \circ C = \mathrm{id}_{K_0}$ as morphisms in $\mathcal{TL}_0$.

By (projective) functoriality of Kh, we have $\mathrm{Kh}(\bar{C}) \circ \mathrm{Kh}(C) = \mathrm{Kh}(\bar{C} \circ C)$, and the remainder of the theorem follows. □

4.3. The concordance homomorphism s. A *Seifert surface* for a knot $K \subset S^3$ is an oriented surface embedded in S^3 whose boundary is K. The 3-ball genus or *Seifert genus* of a knot K, $g_3(K)$, is the minimal genus of a Seifert surface for K:

$$g_3(K) = \min\{g(F) \mid F \hookrightarrow S^3 \text{ with } \partial F = K\}$$

Analogously, the 4-ball genus or *slice genus* of a knot $K \in S^3$ is

$$g_4(K) = \min\{g(F) \mid F \hookrightarrow B^4 \text{ with } \partial F = K\},$$

the minimal genus of a slice surface for K.

concordance homomorphism

A major application of Rasmussen's s invariant in 4D topology is Piccirillo's proof that the Conway knot is not slice; this was a long-standing conjecture until she proved it in less than 8 pages. This is an *Annals of Mathematics* paper. For those of you interested in 4-manifolds and using Kirby calculus, this is a potential final project idea.

5. Some background in category theory and algebra

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