Exercise 4.9. Demonstrate in class Let U denote the group of invertible upper triangular 2×2 matrices

$$\left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{R},\ ad \neq 0 \right\} \subset GL_n(\mathbb{R})$$

and let $\varphi : U \to \mathbb{R}^\times$ be the map that sends $A \mapsto a^2$. Prove that φ is a homomorphism, and determine its kernel and image.

Exercise 4.10. Let $f : \mathbb{R}^+ \to \mathbb{C}^\times$ be the map $f(x) = e^{ix}$. Prove that f is a homomorphism, and determine its kernel and image.

Definition 4.11. Here are some more important vocabulary words:

- A homomorphism $\varphi : G \to G'$ is an **isomorphism** if it is also a set bijection.
- A homomorphism from G to itself ($\varphi : G \to G$) is called an **endomorphism**.
- An *isomorphism* from G to itself is called an **automorphism.**

Remark 4.12. Recall from MAT 108 that there are a couple ways to show that a set map $f : A \to B$ is a bijection.

One way to show that f is bijective is to show that it is both injective and surjective.

- To show that f is injective, you need to show that if $f(a) = f(a')$, then $a = a'$.
- To show that f is surjective, you need to show that for all $b \in B$, there is some $a \in A$ such that $f(a) = b$.

The other way is to exhibit an inverse function $f^{-1} : B \to A$ for f. You need to check that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.

Exercise 4.13. Let $\varphi : G \to H$ be an *isomorphism*. Prove that for all $g \in G$, the order of g is the same as the order of $\varphi(g)$: $|g| = |\varphi(g)|$.

Exercise 4.14. Let G be a group. Prove that the map $\varphi : G \to G$, $x \mapsto x^2$, is an endomorphism of G if and only if G is abelian.

Exercise 4.15. HW03

(a) Let p be a prime number. How many automorphisms does the cyclic group C_p have?

(b) How many automorphisms does C_{24} have?

4.2 Cosets

Before discussing cosets, review equivalence relations/partitions and modular arithmetic.

Definition 4.16. Let H be a subgroup of G, and let $a \in G$. The **left coset** of H containing a is the set

$$aH = \{g \in G \mid g = ah \text{ for some } h \in H\}.$$

Some remarks:

- The set of all left cosets of H in G is $\{bH \mid b \in G\}$. (There are probably repeats!)
- Note that every element $h \in H$ is in the same (left) coset (of H), the **identity** coset, which is the (left) coset of H containing 1. This coset is the set $H \subset G$.

We can also make the same definition for **right cosets**. The right coset of H containing a is

$$Ha = \{g \in G \mid g = ha \text{ for some } h \in H\}.$$

Example 4.17. It's useful to keep a concrete example in mind as a reference. In this example, let $G = \mathbb{Z}$, and let H be the subgroup $3\mathbb{Z}$. Note that the group operation is $+$. We can visualize the cosets of $3\mathbb{Z}$ as the three rows below:

$3\mathbb{Z}$	$\cdots$	-9	-6	-3	0	3	6	9	12	15	$\cdots$
$1+3\mathbb{Z}$	$\cdots$	-8	-5	-2	1	4	7	10	13	16	$\cdots$
$2+3\mathbb{Z}$	$\cdots$	-7	-4	-1	2	5	8	11	14	17	$\cdots$

I like to think of this as an infinite corn-on-the-cob, with the integers spiraling around the cob. In this example, if you break the corn and look at a cross-section, there will be three kernels going around the circle.

Proposition 4.18. Let $H \leq G$. The left cosets of H form a partition of G. (The right cosets of H also form a partition of G.)

Proof. By the definition of the set of left cosets, each coset is nonempty, and the union of all the cosets is G. It remains to check that if two cosets have nonempty intersection, then they are the same coset. It suffices to show that if $a \in bH$, then $aH = bH$.

Suppose $a \in bH$, i.e. there is some $h_a \in H$ such that $a = bh_a$, and therefore also $b = ah_a^{-1}$. Since we want to show a set equivalence, we should check double inclusion:

- ($aH \subseteq bH$) If $ah \in aH$, then $ah = (bh_a)h = b(h_ah) \in bH$.
- ($bH \subseteq aH$) If $bh \in bH$, then $bh = (ah_a^{-1})h = a(h_a^{-1}h) \in aH$.

(The proof for right cosets is nearly identical.) □

Because partitions and equivalence relations are logically the same thing, you can also try proving Proposition 4.18 in terms of equivalence relations.

Exercise 4.19. Prove Proposition 4.18 by defining an equivalence relation on the elements of G such that the equivalence classes agree with the set of left cosets.

Notation 4.20. We will sometimes write G/H to denote the set of left cosets of H. You will see later in this course why this notation both makes sense and also is unfortunate. This is why I keep just writing "the set of left cosets of H in G".

The proof of the following proposition should hopefully give a better sense of how cosets relate to each other.

Proposition 4.21. Let $H \leq G$. All cosets of H (left or right!) have the same cardinality.

Proof. We first show that every left coset has the same cardinality as the identity coset H. Let gH be a left coset of H, and consider the *set map* given by left multiplication by g:

$$\begin{aligned}(g\cdot) : H &\to gH \\ h &\mapsto gh\end{aligned}$$

Because g lives in a group, we automatically get an obvious inverse *set map*

$$\begin{aligned}(g^{-1}\cdot) : gH &\to H \\ x &\mapsto g^{-1}x\end{aligned}$$

(Note that x must necessarily be of the form gh_x for a unique $h_x \in H$, since because if $gh_x = gh'_x$, then by cancellation $h_x = h'_x$. So this map is well-defined.)

Check for yourself that these two maps really are inverse set maps. Therefore g is a bijection, and so H and gH have the same cardinality (by definition of cardinality).

To show that all right cosets have the same cardinality as H (which is both a left and right coset!), use the same trip, but with the set map $(\cdot g) : H \to Hg$, right multiplication by g. □

The following example is a great one to keep in your pocket. Recall that S_3 is the smallest nonabelian group; this makes the cosets behave different from those in abelian groups. Also, S_3 is written multiplicatively, unlike our previous concrete examples.

Example 4.22. *The set of right cosets isn't always the same as the set of left cosets!* As an example, consider $H = S_2 = \langle (12) \rangle$ and $G = S_3$. The left cosets of H are

- $1H = \{1, (12)\}$
- $(13)H = \{(13), (13)(12)\} = \{(13), (123)\}$
- $(23)H = \{(23), (23)(12)\} = \{(23), (132)\}$

whereas the right cosets are

- $1H = \{1, (12)\}$
- $H(13) = \{(13), (12)(13)\} = \{(13), (132)\}$
- $H(23) = \{(23), (23)(13)\} = \{(23), (123)\}$

A group homomorphism $\varphi : G \to G'$ is in particular a set map. Recall from Example 3.20 that the set of subsets $\{\varphi^{-1}(t) \subset G \mid t \in \mathrm{img}(\varphi)\}$ form a partition of G. Because of how well structured groups are, these subsets turn out to exactly be the cosets of the kernel $K = \ker \varphi$!

Remark 4.23. If you were paying attention, you'll notice that I didn't specify whether these were left or right cosets. It turns out that for a special type of subgroup, called a *normal subgroup*, left and right cosets agree. You will also later prove that kernels of homomorphisms are normal.

Proposition 4.24. Let $\varphi : G \to G'$ be a homomorphism, and let $a, b \in G$. Let $K = \ker \varphi$. The following conditions are equivalent (TFAE):

(a) $\varphi(a) = \varphi(b)$

(b) $a^{-1}b \in K$

(c) $b \in aK$

(d) $bK = aK$