4.4 Conjugation, Normal subgroups

Here is a very important definition:

Definition 4.38. Let $g \in G$.

- **Conjugation** by $g \in G$ is the automorphism $c_g : G \to G$ that sends $x \mapsto gxg^{-1}$. (See Exercise 4.28.)
- If $y = gxg^{-1}$, then x and y are **conjugates** of each other. Note that $x = g^{-1}yg$ is obtained by conjugating y by the element g^{-1}.

Exercise 4.39. (Exercise 3.16) HW03 Show that conjugacy is an equivalence relation. The equivalence classes are called **conjugacy classes.**

Exercise 4.40. HW05 Let G be a group, and let $a, b \in G$. Prove that ab and ba are conjugate elements.

Here's another very important definition in this course:

Definition 4.41. A subgroup $H \leq G$ is **normal** if for all $h \in H$, and all $g \in G$, $ghg^{-1} \in H$. If H is a normal subgroup of G, we write $H \trianglelefteq G$.

In other words, a subgroup $H \leq G$ is normal it is closed under conjugation by any element in the whole group G. There are many equivalent ways to say that a subgroup is normal:

Proposition 4.42. Let $H \leq G$. The following are equivalent (TFAE):

(a) H is a normal subgroup of G, i.e. for all $g \in G$ and $h \in H$, we have $ghg^{-1} \in H$.

(b) For all $g \in G$, $gHg^{-1} = H$.

(c) For all $g \in G$, $gH = Hg$.

(d) Every left coset of H in G is also a right coset.

Note: As usual, gHg^{-1} means $\{ghg^{-1} \mid h \in H\}$.

Proof. This is an abridged proof. Make sure you understand the notation gH, Hg, and gHg^{-1} first. Once you are able to work with this kind of notation, the proof of this proposition is quite short.

(a) $\implies$ (b): Suppose $H \trianglelefteq G$. Then for $gHg^{-1} \subset H$ by definition. But $H \subset gHg^{-1}$ as well because $g^{-1}Hg \subset G$.

(b) $\implies$ (a): Now suppose $gHg^{-1} = H$ for all $g \in G$. Let $g \in G$ and $h \in H$. Then $ghg^{-1} \in gHg^{-1} \in H$.

(b) $\iff$ (c) is clear, and (c) $\implies$ (d) is clear.

(d) $\implies$ (c): Suppose every left coset of H is also a right coset, and let $g \in G$. Then gH contains g, and so does Hg, so gH must be the right coset Hg.

□

Remark 4.43. Notice that the proof above was not hard. However, it was important for us to state the proposition as four equivalent statements because we will encounter normal subgroups in a lot of different contexts. Different characterizations will be useful in different contexts.

Exercise 4.44. HW04 Prove that every subgroup of index 2 is a normal subgroup. Show that a subgroup of index 3 need not be normal by exhibiting a counterexample.

Remark 4.45. Here are some immediate observations.

(a) If G is abelian, then any $H \leq G$ is normal.

(b) $\{1\}$ and G are normal in G.

Definition 4.46. The **center** of G, denoted $Z(G)$, is the set of all the elements that commute with every element in G:

$$Z(G) = \{z \in G \mid gz = zg \text{ for all } g \in G\}.$$

We could equivalently define the center to be all the elements that are fixed by conjugation by all elements of G:

$$Z(G) = \{z \in G \mid gzg^{-1} = z \text{ for all } g \in G\}.$$

Kernels of homomorphisms are normal, and this allows us to prove various *isomorphism theorems* later:

Proposition 4.47. If $\varphi : G \to G'$ is a homomorphism, then $\ker \varphi \trianglelefteq G$.

Proof. HW05 □

Exercise 4.48. On the other hand, $\operatorname{img} \varphi$ need not be normal. Prove this by exhibiting a counterexample.

Proposition 4.49.

(a) If $H \leq G$ and $g \in G$, then the set gHg^{-1} is also a subgroup of G.

(b) If G has exactly one subgroup H of order r, then $H \trianglelefteq G$.

Exercise 4.50. Let G be a group of order $|G| = p^r$ where p is prime, and $r \in \mathbb{N}$. Show that G contains a subgroup of order p.

4.5 Aside: Conjugacy classes in S_n

For a permutation $p \in S_n$, the **cycle type** of p is basically the shape of the partition of n that the cycle notation for p creates. This is best described by example:

Example 4.51. The cycle type of $(1\,2)(3\,4\,5)$ in S_7 is $1+1+2+3$, because there are two indices that are fixed (6 and 7), one cycle of size 2, and one cycle of size 3. We usually write the sizes of the blocks in (weakly) ascending order.

Proposition 4.52. The conjugacy classes of S_n are in bijection with the **cycle types**.

Proof. Here is the idea of the proof. Observe that if p sends $i \mapsto j$, then qpq^{-1} sends $q(i) \mapsto q(j)$. So if p has a cycle that looks like

$$(i_1\; i_2\; \cdots\; i_k),$$

then qpq^{-1} has the cycle

$$(q(i_1)\; q(i_2)\; \cdots\; q(i_k)).$$

□

Remark 4.53. Conjugation, whether in the symmetric group or by change-of-basis matrices in linear algebra, is really the algebraic way of describing a *change of perspective.* When we conjugated p by q, all we did was replace the indices in the cycles with their images under q.

Exercise 4.54. Let p and q be permutations in S_n. Prove that pq and qp have cycles of equal sizes.

Exercise 4.55. HW05 Let q be a 5-cycle in S_n, where $n \geq 6$.

(a) What is the cycle type of q^{17}?

(b) In terms of n, how many permutations are there such that $pqp^{-1} = q$?

Exercise 4.56. For each of the following, determine whether σ_1 and σ_2 are conjugate to each other in S_9. If they are conjugate, find a permutation $\tau \in S_9$ such that $\tau\sigma_1\tau^{-1} = \sigma_2$.

(a) $\sigma_1 = (1\,2)(3\,4\,5)$ and $\sigma_2 = (1\,2\,3)(4\,5)$

(b) $\sigma_1 = (1\,3)(2\,4\,6)$ and $\sigma_2 = (3\,5) \circ (2\,4)(5\,6)$

(c) $\sigma_1 = (1\,5)(7\,2\,4\,3)$ and $\sigma_2 = \sigma_1^{2023}$