4.6 Quotient groups

Let $N \trianglelefteq G$. Then $G/N = N\backslash G$, so let's just say *cosets of* N rather than specifying left/right cosets.

Notation 4.55. Let C be a coset of N in G, and say $C = aN$. When we think of C as an element of the set G/N, we may write on of the following:

- $[C] \in G/N$
- $\bar{a} \in G/N$ (i.e. the *equivalence class* of a under the partition by cosets
- $[a]$ (also standard notation for the equivalence class of a)
- abuse notation and write aN, while remembering that we are talking about aN as a single element of a partition, and forgetting about the fact that it's a set itself.

Remark 4.56. Remember that in additive notation, the coset containing a would be written $[C] = \bar{a} = [a] = a + N$.

Just as we have been writing $aN = \{an \mid n \in N\}$, we use similar notation for the product of two subsets of a group G:

Notation 4.57. Let $A, B \subset G$. Then

$$AB = \{x \in G \mid x = ab \text{ for some } a \in A \text{ and } b \in B\} = \{ab \mid a \in A, b \in B\}.$$

Proposition 4.58. *G/N inherits* a group structure from G.

Proof. Define multiplication in G/N by $(aN)(bN) = (ab)N$. Check that this is well-defined, and observe that this is precisely why we need N to be normal. Check that identity and inverses are also preserved. $\square$

Notation 4.59. Let $\overline{G}$ denote the quotient group G/N under the induced multiplication from G. Let $\pi : G \to \overline{G}$ be the obvious map $G \to G/N$. This is called the **canonical map**.

Theorem 4.60. $\pi : G \to \overline{G}$ is a surjective homomorphism whose kernel is N.

Corollary 4.61. Let $a_1, a_2, \ldots, a_k \in G$ such that $\prod_i a_i = 1$. Then $\prod_i \bar{a}_i = \bar{1}$.

Exercise 4.62. Suppose $H \leq G$ is not a normal subgroup. Prove that there exist left cosets aH and bH such that their product $(aH)(bH)$ is not a coset of H.

Quotient groups are intimately related to group homomorphisms via the First Isomorphism Theorem below. This is the first of three *Isomorphism Theorems*, and is the most important one for us.

Theorem 4.63. Let $\varphi : G \to G'$ be a surjective group homomorphism with kernel N. The quotient group $\overline{G} = G/N$ is isomorphic to the image G'. In other words, there is a unique isomorphism $\bar{\varphi} : \overline{G} \to G'$ such that the following diagram *commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G' \\ \downarrow{\scriptstyle\pi} & \nearrow{\scriptstyle\bar{\varphi}} & \\ \overline{G} & & \end{array}$$

Corollary 4.64. Let $\varphi : G \to G'$ be *any* group homomorphism with kernel N and image $H' \leq G'$. Then the quotient group $\overline{G} = G/N$ is isomorphic to t he image H'.

One way we use the First Isomorphism Theorem is to identify quotient groups with more familiar groups that we already know about.

Example 4.65. In the following examples, our groups are abelian, and so every subgroup is normal. For the subgroups-group pairs listed, identify the quotient group as a more familiar group.

(a) $3\mathbb{Z} \leq \mathbb{Z}$, more generally $n\mathbb{Z} \leq \mathbb{Z}$

(b) $\mathbb{R}e_1 \leq \mathbb{R}^2$

(c) $S^1 \subset C^\times$

(d) $\mathbb{R}^+ \subset C^\times$

Example 4.66. Do the same for these nonabelian groups:

(a) $SL_n(\mathbb{R}) \leq GL_n(\mathbb{R})$

(b) $A_n \leq S_n$

Exercise 4.67. Let $H = \{\pm 1, \pm i\} \leq \mathbb{C}^\times$.

(a) Prove that H is normal in $\mathbb{C}^\times$.

(b) Describe explicitly the cosets of H.

(c) Identify the quotient group $\mathbb{C}^\times/H$. (*Hint:* If you're stuck, first play around with the map $\psi : S^1 \times S^1$ given by $e^{i\theta} \mapsto (e^{i\theta})^2$.)

Exercise 4.68. In the general linear group $GL_3(\mathbb{F})$, consider the subsets

$$H = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad K = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $*$ represents an arbitrary element of a field $\mathbb{F}$.

(a) Show that H is a subgroup of $GL_3(\mathbb{F})$. *Hint: First, compute the product*

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Show that K is a *normal* subgroup of H.

(c) For $\mathbb{F} = \mathbb{R}$, identify the quotient group H/K (up to isomorphism). *Hint: Let $A, B \in H$. Under what conditions are A and B in the same coset of K? Use this to construct a surjective homomorphism from H.*

Remark 4.69. The subgroup H discussed here is called the *Heisenberg group*, and we can actually define it using elements of commutative rings, not just fields. This version of this group with $\mathbb{F} = \mathbb{R}$ was used by Weyl to give an algebraic interpretation of Heisenberg's Uncertainty Principle.

Exercise 4.70. Recall that the Klein four group is $V = \{1, a, b, ab\} = \langle a, b \mid a^2 = b^2 = [a, b] = 1\rangle \cong C_2 \times C_2$ (see page 47 in the book).

(a) Prove that the subgroup $N = \{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ in S_4 is isomorphic to the Klein four group.

(b) Prove that N is normal in S_4. *Hint: Use a theorem from lecture on Wednesday; do not use brute force!*

(c) Prove that the subgroup $H = \langle (1\,2), (3\,4)\rangle \leq S_4$ is also isomorphic to V, but is not a normal subgroup of S_4.

(d) Identify the quotient group S_4/N by computing the cosets. *Hint: Recall that $|S_4| = 4! = 24$; use the counting formula. Either define an isomorphism between S_4/N and your candidate group, or define a surjection from S_4 to your candidate group. You do not need to show me that your map is a homomorphism; just check for yourself that it really is.*

(e) How many subgroups are there in S_4 that contain N? (*Do not solve this by brute force!*)

Exercise 4.71. Let $G = (\mathbb{R}^2, +)$ and let $D \leq G$ denote the set of points on the diagonal:

$$D = \{(x, y) \in \mathbb{R}^2 \mid y = x\}.$$

(a) Briefly explain why $D \trianglelefteq G$.

(b) Use the First Isomorphism Theorem to identify the quotient group G/D with a familiar group.