4.7 Product groups

Here are some harder exercises involving normal subgroups that will become useful when we discuss product groups:

Exercise 4.72. HW05 Let K and H be subgroups of a group G.

(a) Prove that the intersection $K \cap H$ is a subgroup of G.

(b) Prove that if $K \trianglelefteq G$, then $K \cap H \trianglelefteq H$.

Exercise 4.73. HW05 Let H and K be subgroups of G.

(a) Prove that if $HK = KH$, then HK is a subgroup of G.

(b) Prove that if H and K are both *normal* subgroups of G, then their intersection $H \cap K$ is also a *normal* subgroup of G.

Definition 4.74. Let $(A, \star)$ and $(B, \diamond)$ be groups. Then $(A \times B, \cdot)$ is a group under the multiplication rule defined by

$$(a_1, b_1)(a_2, b_2) = (a_1 \star a_2, b_1 \diamond b_2)$$

for $a_i \in A$, $b_i \in B$, $i = 1, 2$.

Exercise 4.75. In this exercise, you will verify all the group axioms for $A \times B$.

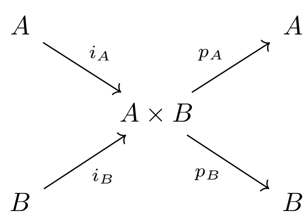
(a) Prove that multiplication is associative.

(b) What's the identity element $A \times B$?

(c) What's the inverse of $(a, b) \in A \times B$?

Exercise 4.76. Prove that $A \times B$ is abelian if and only if both A and B are abelian.

The relationships among the groups A, B, and $A \times B$ is captured by the following maps:



Here i_A and i_B are *injections*; p_A and p_B are *projections*.

(You can look up the definition of these terms, but let's not focus on the nuanced definition of injections and projections in general, for now.)

Example 4.77. $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

The argument for $C_6 \cong C_2 \times C_3$ also works for arbitrary cyclic groups of order rs where $\gcd(r, s) = 1$:

Proposition 4.78. Let r and s be relatively prime integers. A cyclic group of order rs is isomorphic to the product of a cyclic group of order r and a cyclic group of order s.

On the other hand, $C_2 \times C_2$ is not a cyclic group; this is the Klein four group.

While building product groups is easy, it's harder to detect whether a given group is a product of two groups. The last part of the following proposition *characterizes* product groups.

Remark 4.79. Pay attention to the techniques used in the proof; the proof of each statement serves as good practice with normal groups.

Proposition 4.80. Let $H, K \leq G$. let $\mu : H \times K \to G$ be the multiplication map $\mu(h, k) = hk$. Its image is the subset

$$HK = \{hk \mid h \in H, k \in K\} \subset G.$$

(a) μ is injective if and only if $H \cap K = \{1\}$.

(b) μ is a homomorphism from the product *group* $H \times K$ to G if and only if elements of K commute with elements of H: $hk = kh$.

(c) If $H \trianglelefteq G$, then $HK \leq G$.

(d) $\mu : H \times K \to G$ is an isomorphism if and only if

- $H \cap K = \{1\}$
- $HK = G$
- $H, K \trianglelefteq G$.

Proof. See Page 65 in the book, Proposition 2.11.4. □

Remark 4.81. The multiplication map is a set map, a priori. It can even be bijective without being a homomorphism. For example, consider the subgroups $\langle (1\ 2) \rangle$ and $\langle (1\ 2\ 3) \rangle$ inside S_3.

Exercise 4.82. Let G be a group of order 21. Suppose it contains two *normal* subgroups K and N, where $|K| = 3$ and $|N| = 7$. Prove that $G \cong K \times N$.