5.5 Connecting the geometry with the algebra

Question 5.17. Let ℓ be the line of reflection of the isometry $\rho_\theta\tau \in O(2)$. What is the the angle the line ℓ makes with the e_1-axis?

Let's use polar coordinates; we will represent points in the plane as points in the complex plane. Let $re^{i\alpha} \in \mathbb{C}$. Then

$$\rho_\theta\tau(re^{i\alpha}) = \rho_\theta(re^{i\alpha}) = re^{-i\alpha} \cdot e^{i\theta} = re^{i(\theta-\alpha)}.$$

In other words, the reflection $\rho_\theta\tau$ swaps the positions of the two points

$$re^{i\alpha} \leftrightarrow re^{i(\theta-\alpha)}.$$

Hence the angle that the mirror line ℓ makes an angle of

$$\frac{\alpha + (\theta - \alpha)}{2} = \frac{\theta}{2}$$

with the e_1-axis.

Exercise 5.18. Check that the points on the line ℓ are indeed fixed by the reflection $\rho_\theta\tau$.

Question 5.19. Let $g = t_a\rho_\alpha\tau$ be a glide reflection.

(a) What is the angle that the line of reflection makes with the e_1-axis?

(b) What is the **glide vector** v?

Notice that translations do not affect the angle that the line of reflection makes with the horizontal axis. To answer (a), let $\bar{g} = \rho_\alpha\tau$ be the part of g in $O(2)$. *(We will talk more about g vs. $\bar{g}$ when we talk about discrete subgroups of* $\operatorname{Isom}(\mathbb{R}^2)$*.)* By the previous exercise, we know the line of reflection makes an angle of $\alpha/2$ with the e_1-axis.

To answer (b), we first observe that g^2 *is just a translation*, specifically by *twice the glide vector*, $2v$. So we first compute g^2, using our knowledge of the semi-direct product structures of $\operatorname{Isom}(\mathbb{R}^2)$ and $O(2)$:

$$g^2 = (t_a\rho_\alpha\tau)(t_a\rho_\alpha\tau) = t_a(\rho_\alpha\tau)t_a(\rho_\alpha\tau) = t_a t_{\rho_\alpha\tau(a)}(\rho_\alpha\tau)(\rho_\alpha\tau) = t_{a+\rho_\alpha\tau(a)}.$$

Therefore the glide vector for g is $v = \frac{1}{2}(a + \rho_\alpha\tau(a))$.

Exercise 5.20. HW07 Prove that a conjugate of a glide reflection in $\operatorname{Isom}(\mathbb{R}^2)$ is also a glide reflection, and that the glide vectors have the same length.

5.6 Discrete subgroups of $\operatorname{Isom}(\mathbb{R}^2)$

Let $H \leq \operatorname{Isom}(\mathbb{R}^2)$.

- H contains an arbitrarily small translation if, for any $\varepsilon > 0$, there is a translation $t_v \in H$ such that $0 \leq |v| < \varepsilon$.
- Similarly, H contains arbitrarily small rotations if, for any $\varepsilon > 0$, there is a rotation $\rho_\theta \in H$ such that $0 \leq |\theta| < \varepsilon$.

Definition 5.21. A group G of isometries of the plane (i.e. $G \leq \operatorname{Isom}(\mathbb{R}^2)$) is **discrete** if it does not contain arbitrarily small translations or rotations.

In other words, G is **discrete** if there exists a real number ε such that

- if $t_v \in G$ and $v \neq 0$ (i.e. $t_v \neq \text{id}$), then $|v| > \varepsilon$, and
- if $\rho_\theta \in G$, where $\theta \in [-\pi, \pi)$, then $|\theta| \geq \varepsilon$.

Given a discrete group of isometries $G \leq \operatorname{Isom}(\mathbb{R}^2)$, we will study the following subgroups:

- the **translation group** $L \leq G$, a subgroup of the group of translations $T \leq \mathrm{Isom}(\mathbb{R}^2)$
- the **point group** $\overline{G}$, a subgroup of the orthogonal group $O(2) \leq \mathrm{Isom}(\mathbb{R}^2)$.

Exercise 5.22. Explain why, in the setup above, $G \cong L \rtimes \overline{G}$.

The following theorem classifies all possible translation groups:

Theorem 5.23. Every discrete subgroup $L \leq T \cong \mathbb{R}^2$ is one of the following:

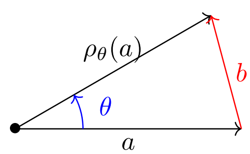
- the zero group: $L = \{0\}$
- the set of integer multiples of a nonzero vector a: $L = \mathbb{Z}a$
- the set of integer combinations of two linearly independent vectors a and b: $L = \mathbb{Z}a + \mathbb{Z}b$. *Groups of this type are called **lattices**.*

The following theorem classifies all possible point groups:

Theorem 5.24 (Crystallographic Restriction)**.** Let Λ be a **discrete** subgroup of $\mathbb{R}^2$, and let $\mathrm{Sym}(\Lambda) \leq \mathrm{Isom}(\mathbb{R}^2)$ denote the group of symmetries of Λ.
Let $H \leq O(2) \cap \mathrm{Sym}(\lambda)$, and suppose that $\Lambda \neq \{0\}$. Then

1. every rotation in H has order 1,2,3,4, or 6, and
2. H is one of the groups C_n or D_n, where $n \in \{1, 2, 3, 4, 6\}$.

Proof. It suffices to prove (a). Let ρ_θ be a rotation in H. Let $a \in \Lambda$ be a *minimal length* translation vector $t_a \in \mathrm{Sym}(\Lambda)$. Then $\rho_\theta t_a = t_{\rho_\theta(a)} \in \mathrm{Sym}(\Lambda)$, so $\rho_\theta(a) \in \Lambda$. Let $b = \rho(a) - a$:



From the figure, we see that $\|b\| < \|a\|$ if $\theta < \pi/3$. So by minimality of a, we must have $\theta \geq \pi/3$. Therefore $|\rho_\theta| \leq 6$.
We can easily construct lattices Λ with symmetries ρ_θ of order $1, 2, 3, 4, 6$. *Try this yourself.*
It remains to show that $\theta = 2\pi/5$ is *impossible*. Let $\phi = 2\pi/5$. If $\rho_\phi \in H$, then $b = \rho_\phi^2(a) + a \in \Lambda$ as well. But then b is shorter than a, which again contradicts the minimality of a. *Use trigonometric functions to prove this for yourself!* □