6 Group actions

Definition 6.1. A **group action** (or **group operation**) is a map $G \times S \to S$, $(g, s) \mapsto g * s$, where G is a group and S is a set, satisfying the following axioms:

(a) $1 * s = s$ for all $s \in S$

(b) (*associative law*) $(gg') * s = g * (g' * s)$ for all $g, g' \in G$ and $s \in S$.

A few remarks:

- We say G *acts on* S, and write $G \curvearrowright S$.
- We often omit the $*$ notation and just write gs for $g * s$. With this notation, the axioms are $1s = s$ and $(gg')s = g(g's)$.
- For each $g \in G$, we get a map $m_g : S \to S$ given by $s \mapsto gs$.

Example 6.2. Let $[n]$ denote the set of indices $\{1, 2, \cdots, n\}$. Then the symmetric group S_n acts on $[n]$.

Here is a more visual example. Let S^2 be the surface of a globe (a *sphere*). We can let $G = S^1, \mathbb{Z}/n\mathbb{Z}$ act on S^2 by rotation about the axis of the globe. Keep this example in mind as we discuss the rest of this section.

Definition 6.3. Let $G \curvearrowright S$, and fix $s \in S$. The **orbit** of s is

$$O_s = \{s' \in S \mid s' = gs \text{ for some } g \in G\} = \{gs \mid g \in G\}.$$

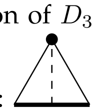
- The orbit of s is the set of elements in S that we can get to by acting on s by an element of g.
- The orbits for a group action are equivalences for the equivalence relation $s \sim s'$ if $s' = gs$ for some $g \in G$.
- Thus, the orbits of the action $G \curvearrowright S$ partition the set S.
- The group acts independently on each orbit.

Example 6.4. Let $G = \text{Isom}(\mathbb{R}^2)$, and let S be the set of triangles in $\mathbb{R}^2$. The orbit of a given triangle T is the set of all triangles congruent (same angles and same side lengths).

Definition 6.5. Let $G \curvearrowright S$, and fix $s \in S$. The **stabilizer** of s is teh set of group elements that leave s fixed:

$$G_s = \{g \in G \mid gs = s\}.$$

This is a subgroup of G. Check this yourself!

Example 6.6. Consider the action of D_3 on an equilateral triangle. Stabilizer of a vertex, an edge, and a perpendicular bisector are all C_2:  The stabilizer of the center of the triangle is the rotation subgroup C_3.

Definition 6.7. Let $G \curvearrowright S$.

- If S consists of one orbit, then the action of G on S is **transitive**.
- If $gs = s$ implies that $g = 1$, then the action of G on S is **free**.

Example 6.8.
- (not free, not transitive) rotation action of $\mathbb{Z}/3\mathbb{Z}$ on the globe
- (not free, transitive) defining action of $\text{Isom}(\mathbb{R}^2)$ on $\mathbb{R}^2$
- (free, not transitive) $H =$ subgroup of $T \leq \text{Isom}(\mathbb{R}^2)$ of horizontal translations

- (free, transitive) action of G on G by left multiplication: $\mu : G \times G \to G$

Remark 6.9. Make sure you're very clear about what set your group is acting on.

Exercise 6.10. It's obvious that the action of G is transitive on each orbit of the action $G \curvearrowright S$. Why?

Proposition 6.11. Let $G \curvearrowright S$, $s \in S$, and $G_s =$ stabilizer of s.

(a) If $a, b \in G$, then $as = bs$ iff $a^{-1}b \in G_s$, iff $b \in aG_s$.

(b) Suppose $s' = as$. Then $G_{s'}$ is a **conjugate subgroup** to G_s:

$$G_{s'} = aG_sa^{-1} = \{g \in G \mid g = aha^{-1} \text{ for some } h \in G_s\}$$

Proof. (a) Clear: $as = bs$ iff $a^{-1}bs = s$.

(b) Show double inclusion.

($G_{s'} \supseteq aG_sa^{-1}$) If $g \in aG_sa^{-1}$, then $g = aha^{-1}$ for some $h \in G_s$. Then $gs' = (aha^{-1})(as) = ahs = as = s'$.

($G_{s'} \subseteq aG_sa^{-1}$) Since $s = a^{-1}s'$, $a^{-1}G_{s'}a \subseteq G_s$ by the same argument.

□

Exercise 6.12. Let $G = GL_n(\mathbb{R})$ act on the set $V = \mathbb{R}^n$ by left multiplication.

(a) Describe the decomposition of V into orbits for this action.

(b) What is the stabilizer of e_1?

(c) Is this action of G on $V - \{0\}$ free, transitive, both, or neither?

Exercise 6.13. Does the rule $P * A = PAP^\top$ define an operation of GL_n on $M_{n\times n}$, the set of $n \times n$ matrices? *Here, $P^\top$ is the transpose of the matrix $P \in GL_n$.*