- (free, transitive) action of G on G by left multiplication: $\mu : G \times G \to G$

Remark 6.9. Make sure you're very clear about what set your group is acting on.

Exercise 6.10. It's obvious that the action of G is transitive on each orbit of the action $G \curvearrowright S$. Why?

Exercise 6.11. Suppose a group G **acts freely** on a set S (i.e. the group action $G \curvearrowright S$ is free). Prove that for any $s \in S$, the stabilizer G_s is the trivial subgroup of G.

Proposition 6.12. Let $G \curvearrowright S$, $s \in S$, and $G_s =$ stabilizer of s.

(a) If $a, b \in G$, then $as = bs$ iff $a^{-1}b \in G_s$, iff $b \in aG_s$.

(b) Suppose $s' = as$. Then $G_{s'}$ is a **conjugate subgroup** to G_s:

$$G_{s'} = aG_s a^{-1} = \{g \in G \mid g = aha^{-1} \text{ for some } h \in G_s\}$$

Proof. (a) Clear: $as = bs$ iff $a^{-1}bs = s$.

(b) Show double inclusion.

$(G_{s'} \supseteq aG_s a^{-1})$ If $g \in aG_s a^{-1}$, then $g = aha^{-1}$ for some $h \in G_s$. Then $gs' = (aha^{-1})(as) = ahs = as = s'$.

$(G_{s'} \subseteq aG_s a^{-1})$ Since $s = a^{-1}s'$, $a^{-1}G_{s'}a \subseteq G_s$ by the same argument.

□

Exercise 6.13. HW09 Does the rule $P * A = PAP^\top$ define an operation of GL_n on $M_{n\times n}$, the set of $n \times n$ matrices? *Here, $P^\top$ is the transpose of the matrix $P \in GL_n$.*

Exercise 6.14. HW09 Let $G = GL_n(\mathbb{R})$ act on the set $V = \mathbb{R}^n$ by left multiplication.

(a) Describe the decomposition of V into orbits for this action.

(b) What is the stabilizer of e_1?

(c) Is this action of G on $V - \{0\}$ free, transitive, both, or neither?

6.1 The action of G on cosets of $H \leq G$

Let H be a subgroup of G (not necessarily normal). Then G acts on the set of left cosets G/H of H in a *natural* way, i.e. in an obvious or canonical way:

$$g * [aH] = [gaH].$$

Observe that

- This action is transitive. Why?
- The stabilizer of the coset $[H]$ is the subgroup H. Why?

Example 6.15. Let $G = D_3 = \{1, \rho, \rho^2, \tau, \rho\tau, \rho^2\tau\}$, and let $H = \langle\tau\rangle = \{1, \tau\}$.
The left cosets of H are

$$H = \{1, \tau\} \qquad \rho H = \{\rho, \rho\tau\} \qquad \rho^2 H = \{\rho^2, \rho^2\tau\}$$

To understand how G acts on G/H, we just need to know how the generators ρ and τ act on G/H. (Why?)

- $\rho*$ sends $H \mapsto \rho H \mapsto \rho^2 H \mapsto H$
- $\tau*$ sends $H \mapsto H$, $\rho H \leftrightarrow \rho^2 H$

In other words, if we label the three cosets $H, \rho H, \rho^2 H$ as C_1, C_2, C_3 respectively, then the action of ρ is $(1\ 2\ 3)$ on the indices, and the action of τ is $(2\ 3)$ on the indices.

6.2 Orbit-stabilizer theorem

Proposition 6.16. Suppose a group G acts on a set S. Let $s \in S$. Let G_s denote the stabilizer of s, and let O_s denote the orbit of s.

There is a bijective map (of sets!)

$$\begin{aligned} \varepsilon : G/G_s &\to O_s \\ [aG_s] &\mapsto as \end{aligned}$$

that respects the action of G on both sides, i.e.

$$\varepsilon(g[C]) = g\varepsilon([C])$$

for every coset C and every element $g \in G$. (We say that the map ε is G-**equivariant**.)

Proof. For the purposes of this proof, we let $H = G_s$.

First, we need to show that ε is well-defined. Suppose $aH = bH$; we need to show that $as = bs$. Since $a \in bH$, there is some $h \in H$ such that $a = bh$. Since $h \in H = G_s$ fixes s, $as = bhs = bs$.

Second, we show that ε is injective. If $\varepsilon(aH) = \varepsilon(bH)$, then $as = bs$, so $b^{-1}as = b^{-1}bs = 1s = s$. Then $b^{-1}a \in H$, so $aH = bH$ indeed.

Third, we show that ε is surjective. If $s' \in O_s$, then there is some $g \in G$ such that $s' = gs$. Then $\varepsilon(gH) = gs = s'$.

Finally, we need to check that ε is G-equivariant. Let $g \in G$, and let $[aH] \in G/H$. Then

$$\varepsilon(g[aH]) = \varepsilon([gaH]) = gas = g(as) = g\varepsilon([aH]).$$

□

Exercise 6.17. Exhibit the bijective map ε from the orbit-stabilizer theorem explicitly, for the case where G is the dihedral group D_4 and S is the set of vertices of a square.

The Orbit-Stabilizer Theorem is very often used to count things. Recall that the Counting Formula tells us

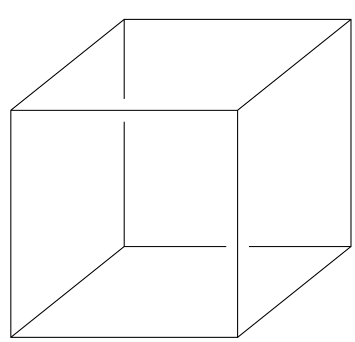
$$|G| = |H||G/H|.$$

In terms of group actions, we have yet another version of the counting formula.

Observation 6.18 (Counting Formula)**.** Let S be a finite set on which G acts. Let $s \in S$. By the Orbit-Stabilizer Theorem,

$$|G| = |G_s||O_s|.$$

Here is an example that illustrates we can use this formula to determine the size of a symmetry group. Consider a cube:



Question 6.19. How big is the set of orientation-preserving symmetries of the cube?

First, a couple of remarks:

1. To rephrase this in terms of abstract algebra, we first note that the set of symmetries is actually a group. So, we can rephrase this question as follows. Let G be the group of orientation-preserving symmetries of the cube. What is $|G|$, the order of the group G?
2. This the 3D analogue to the symmetries of plane figures, such as a square. The symmetries must be isometries of $\mathbb{R}^3$.
3. **Orientation-preserving** means that you can't reflect the cube through a plane; we really want to only consider symmetries that you can physically perform on a real-life cube, such as a die.
4. If we're looking at a solid object in real life (i.e. not an infinite 3D object), then the group of orientation-preserving symmetries consists only of rotations. So, the book will call these **rotational symmetries**

In order to answer this, one could try to count all the symmetries. Or, one could focus on, say, the set of faces. That is, there is clearly a natural action of G on the set of 6 faces of a cube. Let f be a particular face. The only actions I can perform that preserve a given face are the four rotations about the line normal to that face. Therefore $|G_f| = 4$. The orbit of f is all six faces of the cube, so $|O_f| = 6$. Then by the Orbit-Stabilizer Theorem and Counting Formula, we know $|G| = 24$.

Exercise 6.20. Let G be the set of rotational symmetries of a regular dodecahedron. This is a solid with 12 faces that are all regular pentagons. What is $|G|$?

We can also use algebra to figure out the size of a set that a group acts on, by using the following observation.

Observation 6.21 (Decomposition of S into orbits)**.** Let S be a finite set on which G acts, and let $O_1, O_2, \ldots, O_k$ be the set of orbits. Then

$$|S| = |O_1| + |O_2| + \cdots + |O_k|.$$

This observation will become more important later when we study conjugacy classes of groups.