## 6.5 $p$-groups

The conjugation action of $G \curvearrowright G$ is one tool that mathematicians have used to classify finite groups. We will now discuss the simplest cases, which are groups of order a prime power.

Let $p$ be a prime number. Recall that we were able to completely classify groups of order $p$: they are all isomorphic to $C_p$. Review exercise: Can you prove this now?

**Definition 6.37.** A $p$-**group** is a group of order $p^r$ where $r \geq 1$.

**Proposition 6.38.** The center of a $p$-group is not the trivial group.

*Proof.* Suppose $|G| = p^r$ and consider the class equation. All conjugacy classes must be of order $p$, but we know $|C_1| = 1$. So there must be at least $p$ one's, i.e. at least $p$ elements in the center. □

**Exercise 6.39.** Use a similar argument to prove the following theorem:

**Theorem 6.40.** Let $G$ be a $p$-group, and suppose $G$ acts on a finite set $S$. If the order of $S$ is not divisible by $p$, then there is a **fixed point** of the action $G \curvearrowright S$, i.e. an element $s \in S$ where $G_s = G$.

**Proposition 6.41.** Every group of order $p^2$ is abelian.

*Proof.* Let $|G| = p^2$. By Proposition 6.38, $Z(G) \neq \{1\}$. A priori there are two possibilities: $|Z(G)|$ is either $p^2$ or $p$. If $|Z(G)| = p^2$, then we are done. We will now show that $|Z(G)|$ *cannot* be $p$.

By way of contradiction, suppose that $|Z(G)| = p$. Now let $x \in G - Z(G)$. Then $Z(x)$ contains $\langle x \rangle$ and also $Z(G)$, so the order of $Z(x)$ must be $p^2$. But then $x \in Z(G)$, which is a contradiction. □

**Corollary 6.42.** A group of order $p^2$ is either cyclic, or the product of two cyclic groups of order $p$.

*Proof.* Pick an element $x \neq 1$ in $G$. Then $|x| \in \{p, p^2\}$. If $|x| = p^2$, then $G = \langle x \rangle \cong C_{p^2}$.

If $|x| = p$, then pick some $y \notin \langle x \rangle$. Now observe: Can you prove each of these?

- $\langle x \rangle, \langle y \rangle \trianglelefteq G$
- $\langle x \rangle \cap \langle y \rangle = \{1\}$
- $\langle x \rangle \langle y \rangle = G$

Therefore $G \cong \langle x \rangle \times \langle y \rangle$. □

**Remark 6.43.** We don't have the tools to prove the following fact, but it's an important theorem in algebra:

**Theorem 6.44** (Classification of finite abelian groups)**.** If $G$ is a finite abelian group, then it is isomorphic to a product of finite cyclic groups.

More generally, there is a related theorem for *finitely generated* abelian groups. The proof relies on thinking about abelian group as $\mathbb{Z}$-*modules*, which you'll learn about later in the 150 series.