MAT 150A HW01

[add your name here]

Due Tuesday, 1/16/24 at 11:59 pm on Gradescope

Instructions Solve the following problems, and then type up your solutions in full sentences after the

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\solution
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command following each exercise. It may help to look at how I typed the exercise, e.g. to learn the command used to typeset a particular symbol. Compile often. See the instructions in HW00 if you're unsure how to use Overleaf.

Proof-based course This is a proof-based course and you are expected to **clearly prove** all your claims. If you're wondering how much detail to include, a good rule of thumb is that your proofs should be slightly more detailed than the proofs in the book, but not less detailed. They should also not be unreasonably verbose.

Reminder Homeworks must be typed using LaTeX **in full sentences with proper mathematical formatting**. Handwritten homeworks will not be accepted. If there is a documented reason why you can't type up your homework, let me know and we can discuss an alternate policy. Otherwise, please consider learning how to properly write and typeset mathematics as part of this course.

The following resources may be useful as you learn to use TeX and Overleaf:

- Overleaf's introduction to LaTeX:
  https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes
- Detexify:
  https://detexify.kirelabs.org/classify.html

Grading Most (parts of) problems will be graded for completion out of 5 points. A few selected problems will be graded more carefully; these will be revealed after the homework is due. Abridged solutions will be posted after the 24-hour grace period after the homework due date.

Greek letters In mathematics (and many sciences), we require many more symbols than the English alphabet. Greek letters are used very often, so please familiarize yourself with the Greek alphabet if you haven't seen symbols like ρ or τ before:

https://en.wikipedia.org/wiki/Greek_alphabet

In LaTeX, the command for a Greek letter (capital or lowercase) is always a backslash followed by the English spelling of that letter. For example, capital *Gamma* and lower case *gamma* are written Γ and γ, respectively.

Exercise 1

Let G be a group. Since every element in the group has an inverse, we can "do algebra" in the group, in the sense that we can solve equations:

Proposition. (Cancellation Law) Let $a, b, c \in G$. If $ab = ac$ or $ba = ca$, then $b = c$. If $ab = a$ or if $ba = a$, then $b = 1$.

Let $x, y, z, w \in G$.

(a) Solve for y, given that $xyz^{-1}w = 1$.

(b) Suppose that $xyz = 1$. Does it follow that $yzx = 1$? Does it follow that $yxz = 1$?

SOLUTION.

(a)

(b)

Exercise 2

Let's recall how remainders show up when dividing integers. Since

$$32 = 6 \cdot 5 + 2,$$

we say that *32 divided by 5 is 6 remainder 2*. In modular arithmetic, we only care about the remainder, 2. Another way this is commonly written is

$$32 \equiv 2 \mod 5 \qquad \text{or} \qquad 32 \equiv 2 \pmod 5.$$

(a) Recall that, as a set, $\mathbb{Z}/5\mathbb{Z} = \{0, 1, 2, 3, 4\}$. Fill out the rest of the addition table provided below. This is called the *group table* for the group $(\mathbb{Z}/5\mathbb{Z}, +)$.

(b) Let $\mathbb{Z}/5\mathbb{Z}^\times$ denote the set $\{1, 2, 3, 4\} = \mathbb{Z}/5\mathbb{Z} - \{0\}$. It turns out that we can define a multiplication group operation $\bullet$ using remainders as well. That is, for $x, y \in \mathbb{Z}/5\mathbb{Z}^\times$, $x \bullet y$ is just $xy \pmod 5$ (where x and y are first multiplied as integers). For example, $3 \cdot 4 = 12$, and 12 modulo 5 is 2, so $3 \bullet 4 = 2$ in $\mathbb{Z}/5\mathbb{Z}^\times$.

Fill out the group table for the group $(\mathbb{Z}/5\mathbb{Z}^\times, \bullet)$ provided below.

SOLUTION.

(a) Group table for $(\mathbb{Z}/5\mathbb{Z}, +)$:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	?	?	?	?	?
3	?	?	?	?	?
4	?	?	?	?	?

(b) Group table for $(\mathbb{Z}/5\mathbb{Z}^{\times}, \bullet)$:

$\bullet$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	?	?	?	?
4	?	?	?	?

Exercise 3

Prove that if $x^2 = 1$ for all $x \in G$, then G is abelian.

SOLUTION.

Exercise 4

For a real number $r \in \mathbb{R}$, the **floor** of r, denoted $\lfloor r \rfloor$, is the greatest integer less than or equal to r. For example, $\lfloor 0.3 \rfloor = 0$, $\lfloor 65.\bar{3} \rfloor = 65$, $\lfloor \pi \rfloor = 3$, and $\lfloor -0.8 \rfloor = -1$.

Let $G = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$, and for $x \in G$, let $x \star y$ be the fractional part of the real number $x + y$, i.e.

$$x \star y = x + y - \lfloor x + y \rfloor.$$

(a) Prove that $(G, \star)$ is a group, and that it is abelian.

(b) Prove that $(G, \star)$ is **not** a cyclic group. (Hint: Use cardinality.)

Remark. This group G is known as the *real numbers modulo 1*, in analogy with the construction of $\mathbb{Z}/n\mathbb{Z}$, the integers modulo n.

SOLUTION.

(a)

(b)

Exercise 5

Let G be a group, and let $g_1, g_2, \ldots, g_n$ be a n elements of G. Prove that

$$(g_1 g_2 \cdots g_n)^{-1} = g_n^{-1} g_{n-1}^{-1} \cdots g_2^{-1} g_1^{-1}.$$

Hint: First prove this holds for $n = 2$, and then use induction.

SOLUTION.