

MAT 150A HW02

[add your name here]

Due Tuesday, 1/23/24 at 11:59 pm on Gradescope

Instructions Solve the following problems, and then type up your solutions in full sentences after the

`\solution`

command following each exercise. It may help to look at how I typed the exercise, e.g. to learn the command used to typeset a particular symbol. Compile often. See the instructions in HW00 if you're unsure how to use Overleaf.

Proof-based course This is a proof-based course and you are expected to **clearly prove** all your claims. If you're wondering how much detail to include, a good rule of thumb is that your proofs should be slightly more detailed than the proofs in the book, but not less detailed. They should also not be unreasonably verbose.

Reminder Homeworks must be typed using LaTeX **in full sentences with proper mathematical formatting**. Handwritten homeworks will not be accepted. If there is a documented reason why you can't type up your homework, let me know and we can discuss an alternate policy. Otherwise, please consider learning how to properly write and typeset mathematics as part of this course.

The following resources may be useful as you learn to use TeX and Overleaf:

- Overleaf's introduction to LaTeX:
https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes
- Detexify:
<https://detexify.kirelabs.org/classify.html>

Grading Most (parts of) problems will be graded for completion out of 5 points. A few selected problems will be graded more carefully; these will be revealed after the homework is due. Abridged solutions will be posted after the 24-hour grace period after the homework due date.

Greek letters In mathematics (and many sciences), we require many more symbols than the English alphabet. Greek letters are used very often, so please familiarize yourself with the Greek alphabet if you haven't seen symbols like ρ or τ before:

https://en.wikipedia.org/wiki/Greek_alphabet

In LaTeX, the command for a Greek letter (capital or lowercase) is always a backslash followed by the English spelling of that letter. For example, capital *Gamma* and lower case *gamma* are written Γ and γ , respectively.

Exercise 1

- (a) Prove that the transpose of a permutation matrix is its inverse.
- (b) Prove that the determinant of a permutation matrix is always ± 1 .

Definition. A permutation $p \in S_n$ is **even** if the determinant of the corresponding permutation matrix P is 1; the permutation is **odd** if the determinant of P is -1.

- (c) Let $p \in S_n$, and write p as a composition (or equivalently, product) of k transpositions:

$$p = \tau_{i_1} \circ \tau_{i_2} \circ \dots \circ \tau_{i_k}$$

Prove that p is even if and only if k is even, and that p is odd if and only if k is odd.

SOLUTION.

Exercise 2

- (a) Prove that $\mathbb{C}^\times = \mathbb{C} - \{0\}$ is a group under complex multiplication. *Hint: Use polar coordinates, i.e. $z = re^{i\theta}$.*
- (b) Find a *representation* of \mathbb{C}^\times in $GL_2(\mathbb{R})$. That is, assign every element $z \in \mathbb{C}^\times = \mathbb{C} - \{0\}$ to a 2×2 invertible matrix so that matrix multiplication agrees with multiplication in \mathbb{C}^\times (and prove that they indeed agree). *Hint: Use rectangular coordinates, i.e. $z = x + iy$.*

SOLUTION.

Exercise 3

Prove the Subgroup Criterion:

Proposition. A subset H of a group G is a subgroup if and only if $H \neq \emptyset$ and for all $a, b \in H$, $ab^{-1} \in H$.

SOLUTION.

Exercise 4

Let $p = (1\ 2)(3\ 4\ 5)$ and $q = (1\ 2\ 3)(4\ 5)$, where $p, q \in S_n$ for some $n \geq 5$.

- (a) Compute pq and qp .
- (b) What is $|p|$? What is $|p^{2024}|$?

SOLUTION.

Exercise 5

- (a) Let $a, b \in G$. Prove that $|ab| = |ba|$.
- (b) Show by example that the product of elements of finite order in a group need not have finite order.
- (c) Show that if G is abelian, then the product of elements of finite order has finite order.

SOLUTION.