MAT 150A HW04

Solutions

Due Tuesday, 2/4/24 at 11:59 pm on Gradescope

Instructions Solve the following problems, and then type up your solutions in full sentences after the

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\solution
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command following each exercise. It may help to look at how I typed the exercise, e.g. to learn the command used to typeset a particular symbol. Compile often. See the instructions in HW00 if you're unsure how to use Overleaf.

Proof-based course This is a proof-based course and you are expected to **clearly prove** all your claims. If you're wondering how much detail to include, a good rule of thumb is that your proofs should be slightly more detailed than the proofs in the book, but not less detailed. They should also not be unreasonably verbose.

Reminder Homeworks must be typed using LaTeX **in full sentences with proper mathematical formatting**. Handwritten homeworks will not be accepted. If there is a documented reason why you can't type up your homework, let me know and we can discuss an alternate policy. Otherwise, please consider learning how to properly write and typeset mathematics as part of this course.

The following resources may be useful as you learn to use TeX and Overleaf:

- Overleaf's introduction to LaTeX:
  https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes
- Detexify:
  https://detexify.kirelabs.org/classify.html

Grading Most (parts of) problems will be graded for completion out of 5 points. A few selected problems will be graded more carefully; these will be revealed after the homework is due. Abridged solutions will be posted after the 24-hour grace period after the homework due date.

Midterm Exam 1 Exam 1 will be this Friday, February 2, 2024, in class. This is a traditional pen-and-paper exam. You will be expected to write in full sentences with proper mathematical formatting. Style will count for a portion of each problem's point value.

The material tested corresponds to Lectures 1–8 and HW01–HW03. The relevant book sections are §2.1–2.7, 2.9, plus the dihedral group (§6.5) and group presentations (§7.10). Key terms include but are not limited to groups, generators and relations, subgroups, homomorphisms, isomorphisms, kernels, images, order of a group, order of an element, the symmetric group, cyclic groups, additive/multiplicative notation, complex numbers.

The exercises labeled "Exam 1 Review" below are in the style of exam problems and cover material that could be on Exam 1. Note that solutions will not be provided until after the exam. If you're wondering how much you should write in your solutions on the exam, look at previous homework solutions.

Exercise 1

(Exam 1 Review) Let G be a group. Prove that the map $\varphi : G \to G$, $x \mapsto x^2$, is an endomorphism of G if and only if G is abelian.

SOLUTION.

Exercise 2

(Exam 1 Review) Recall that the Klein four group is the abelian group $V = \{1, a, b, c\}$ where $a^2 = b^2 = c^2 = 1$ and $c = ab$.

(a) Write down a presentation for the group V.

(b) Find an injective homomorphism $\varphi : V \to S_4$, where S_4 is the symmetric group on 4 letters.

Solution.

Exercise 3

(a) Let p be a prime number. How many automorphisms does the cyclic group C_p have?

(b) How many automorphisms does C_{24} have?

Solution.

Exercise 4

Recall that S_4 is the group of all permutations of $\{1, 2, 3, 4\}$. View $S_3 \leq S_4$ as the subgroup of all permutations that fix 4 but permute $\{1, 2, 3\}$.

(a) How many elements are in each left coset of S_3? How many left cosets are there?

(b) Explicitly write down all the left cosets of S_3 in S_4 as subsets of S_4. *Use cycle notation, and simplify your cycle notation so that each index appears only once. For example, rewrite* $(14)(12)$ *as* (124). *This will help show that you have actually formed a partition of* S_4.

Solution.

Exercise 5

Let $\varphi : G \to G'$ be a group homomorphism. Suppose that $|G| = 18$ and $|G'| = 15$, and that φ is not the trivial homomorphism. What is the $|\ker \varphi|$?

Solution.

Exercise 6

Prove that every subgroup of index 2 is a normal subgroup, and show by example that a subgroup of index 3 need not be normal.

Solution.