MAT 150A HW06

[add your name here]

Due Tuesday, 2/20/24 at 11:59 pm on Gradescope

Proof-based course This is a proof-based course and you are expected to **clearly prove** all your claims. If you're wondering how much detail to include, a good rule of thumb is that your proofs should be slightly more detailed than the proofs in the book, but not less detailed. They should also not be unreasonably verbose.

Reminder Homeworks must be typed using LaTeX **in full sentences with proper mathematical formatting**. Handwritten homeworks will not be accepted. If there is a documented reason why you can't type up your homework, let me know and we can discuss an alternate policy. Otherwise, please consider learning how to properly write and typeset mathematics as part of this course.

Exercise 1

Let H and K be subgroups of G. Prove that if $HK = KH$, then HK is a subgroup of G.

SOLUTION.

Exercise 2

(a) Let C be a cyclic group generated by an element $g \in C$. Let G be another group. Prove that any homomorphism $\varphi : C \to G$ is determined by the value of φ on g.

(b) How many homomorphisms are there from $\mathbb{Z} \to \mathbb{Z}/15\mathbb{Z}$?

(c) How many homomorphisms are there from $\mathbb{Z}/15\mathbb{Z} \to \mathbb{Z}$?

SOLUTION.

Exercise 3

Use the First Isomorphism Theorem to find an isomorphism between G/H and a more familiar group.

(a) $G = \mathbb{R}^2$, $H = \mathbb{R}e_1$, where e_1 is the first standard basis vector.

(b) $G = \mathbb{C}^\times$, $H = \mathbb{R}^\times_{>0}$

SOLUTION.

Exercise 4

Recall that the Klein four group is $V = \{1, a, b, ab\} = \langle a, b \mid a^2 = b^2 = [a, b] = 1\rangle \cong C_2 \times C_2$ (see page 47 in the book).

(a) Prove that the subgroup $N = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ in S_4 is isomorphic to the Klein four group.

(b) Prove that N is normal in S_4. *Hint: Do not use brute force!*

(c) Prove that the subgroup $H = \langle(1\ 2), (3\ 4)\rangle \leq S_4$ is also isomorphic to V, but is not a normal subgroup of S_4.

(d) Identify the quotient group S_4/N. That is, use the First Isomorphism Theorem to find an isomorphism from S_4/N to a more familiar group. *Hint: Use the counting formula first to figure out what the order of the quotient group is.*

(e) How many subgroups are there in S_4 that contain N? *Hint: Do not use brute force!*

Solution.

Exercise 5

Let G be a group of order 21. Suppose it contains two **normal** subgroups K and N, where $|K| = 3$ and $|N| = 7$. Prove that $G \cong K \times N$.

Solution.