(lecture 4)

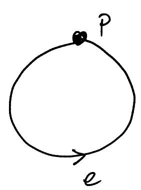
START

Examples: (Compare + Contrast)

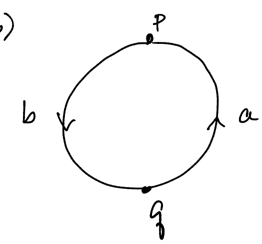
① S^1, w/ diff't Δ-cpx structures

Let X in all cases denote the space w/ the Δ-structure.

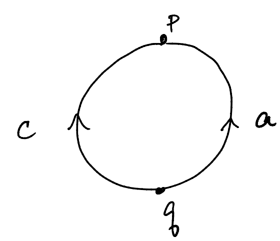
(a)



(b)



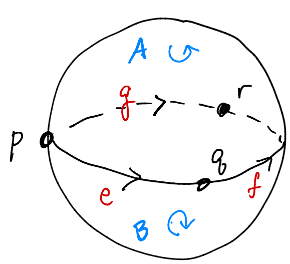
(c)



(check that these really are Δ-cpx structures...)

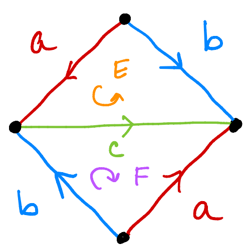
② S^2 vs $\mathbb{RP}^2$ Check these are Δ-cpxs.

(a)



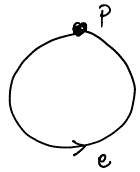
orientable

(b)



non orientable!

$\boxed{\text{eg 1 a}}$



$\Delta_0 = \mathbb{Z}\sigma_0$ where $\sigma_0 : \Delta^0 \longrightarrow X$, $[v_0] \longmapsto p$

$\Delta_1 = \mathbb{Z}\sigma_1$ where $\sigma_1 : \Delta^1 \longrightarrow X$, $[v_0, v_1] \rightsquigarrow e$

homeo

$$0 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Only ∂_1 is nontrivial, determined by how it maps basis element σ_1:

$$\partial_1 \sigma_1 = (-1)^0 \sigma_1|_{[v_0]} + (-1)^1 \sigma_1|_{[v_1]} = \sigma_0 - \sigma_0 = 0.$$

So our chain complex is

$$\begin{array}{ccccccc} 0 \longrightarrow & C_1 = \mathbb{Z}\sigma_1 & \longrightarrow & C_0 = \mathbb{Z}\sigma_0 & \longrightarrow 0 \\ & \sigma_1 & \longmapsto & 0 & \end{array}$$

More abstractly:

$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \underline{\mathbb{Z}} \xrightarrow{0} 0$$

underline where C_0 is ("homological degree 0")

Take homology:

$$H_0^\Delta(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\mathbb{Z}\sigma_0}{\{0\}} \cong \mathbb{Z}.$$

$$H_1^\Delta(X) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \mathbb{Z}\sigma_1 / \{0\} \cong \mathbb{Z}$$

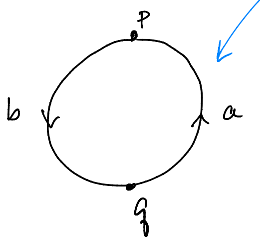
Therefore the simplicial homology of X is

$$H_n^\Delta(X) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$\boxed{\text{eg 1b}}$

notation:
$\sigma_a : \Delta^1 \to X$

$X =$



$$0 \longrightarrow C_1 = \mathbb{Z}\langle a,b\rangle \longrightarrow C_0 = \mathbb{Z}\langle p,q\rangle \longrightarrow 0$$

$$a \longmapsto p-q$$
$$b \longmapsto q-p$$

actually:
$$\sigma_a \longmapsto \sigma_p - \sigma_q$$
$$\sigma_b \longmapsto \sigma_q - \sigma_p$$

$*$ Abusing notation: writing α for σ_α.

- $\ker \partial_0 = \mathbb{Z}\langle p,q\rangle$

  $\operatorname{im} \partial_1 = \mathbb{Z}\langle p-q\rangle$

  p is homologous to q

  So they are in the same equiv. class in homology (a subquotient group)

  $\Rightarrow H_0^\Delta(X) = \mathbb{Z}\langle p,q\rangle / \langle p-q\rangle = \mathbb{Z}\langle [p]\rangle \cong \mathbb{Z}$

  $[p]=[q]$

- $\ker \partial_1 = ??? = \mathbb{Z}\langle a+b\rangle$

  $\operatorname{im} \partial_1 = 0$

  $\Rightarrow H_1^\Delta(X) = \mathbb{Z}\langle p+q\rangle \cong \mathbb{Z}.$

- Thus $H_i^\Delta(X) = \begin{cases} \mathbb{Z} & i=0,1 \\ 0 & \text{o/w} \end{cases}$

$*$ Same as eg 1a!

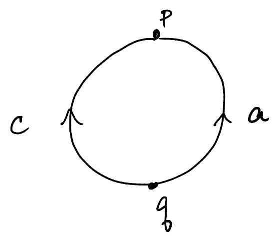
More definitions (Important)

- An element of some C_i is called a chain.
- An element of some $\ker \partial_i$ is called a cycle
- An element of some $\operatorname{im} \partial_i$ is called a boundary.
- Two cycles $x,y \in \ker \partial_i \subset C_i$ homologous if they differ by a boundary, ie $y-x \in \operatorname{im} \partial_{i+1}$.

We will now write a bit less.

$\boxed{eg\ 1c}$

$X =$



$\Delta_0 = \mathbb{Z}\langle p, q\rangle$

$\Delta_1 = \mathbb{Z}\langle a, c\rangle$

$\partial_1 a = p - q$

$\partial_1 c = p - q$

$$H_0^\Delta(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \mathbb{Z}\langle p, q\rangle / \langle p - q\rangle \cong \mathbb{Z}$$

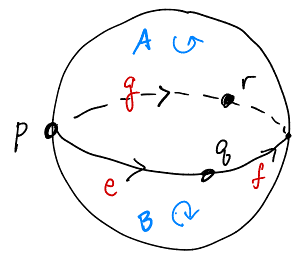
$$H_1^\Delta(X) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \mathbb{Z}\langle a - c\rangle / 0 \cong \mathbb{Z}$$

Same homology.

We've shown in 3 ways that $H_*(S^1)$ is

$\mathbb{Z}$ in dims 0 and 1, and 0 otherwise.

$\boxed{\text{eg 2a}}$

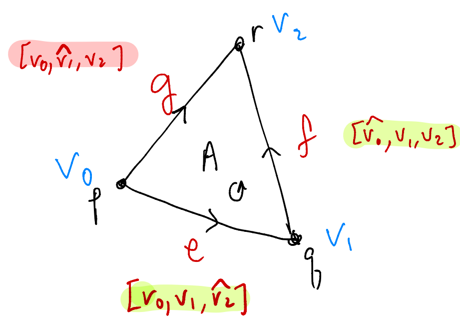
$X =$ 

$$0 \to \mathbb{Z}\langle A,B\rangle \longrightarrow \mathbb{Z}\langle e,f,g\rangle \longrightarrow \mathbb{Z}\langle p,q,r\rangle \longrightarrow 0$$

- Boundary maps

$\partial_1 e = q-p \quad \partial_1 f = r-q \quad \partial_1 g = r-p$

$\partial_2 A = e-g+f$



* You can also get this by comparing ∂ orientation of ∂A and the simplicial orientation; see HW01 for Δ^3 example.

Similarly $\partial B = e-g+f$

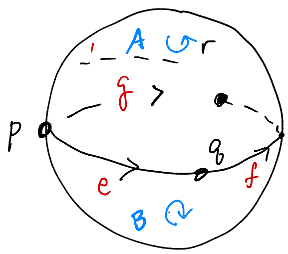
- Homology

$$H_0^{\Delta}(X) = \ker\partial_0 / \operatorname{im}\partial_1 = \mathbb{Z}\langle p,q,r\rangle / \langle \underbrace{q-p, \ r-q, \ r-p}_{\substack{r-p = (q-p)+(r-q) \\ \text{linear dependence!}}} \rangle$$

change basis:

$$= \mathbb{Z}\langle p-q, q-r, r\rangle / \langle q-p, r-q\rangle \cong \mathbb{Z}\langle r\rangle \cong \mathbb{Z}.$$

TBC'd either Tuesday Disc. recorded or on HW on your own!



$$H_1^\Delta(X) = \ker \partial_1 / \operatorname{img} \partial_2 = \underbrace{\mathbb{Z}\langle e-g+f\rangle} / \mathbb{Z}\langle e-g+f\rangle \cong 0.$$

this is non obvious!
take a moment to confirm

$$H_2^\Delta(X) = \ker \partial_2 / \operatorname{im} \partial_3 = \mathbb{Z}\langle A-B\rangle / 0 \cong \mathbb{Z}$$

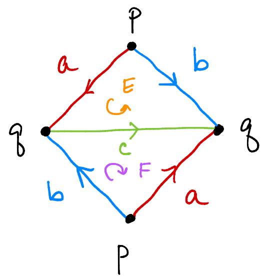
- Therefore

$$H_n^\Delta(X) \cong \begin{cases} \mathbb{Z} & n=0,2 \\ 0 & o/w. \end{cases}$$

* compare w/ $H_*(S^1)$.

Discussion 2 / Lecture 5, part Ⓘ

$\boxed{\text{eg 2b}}$



$$0 \to \mathbb{Z}\langle E,F\rangle \xrightarrow{\partial_2} \mathbb{Z}\langle a,b,c\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle p,q\rangle \to 0$$

$$E \longmapsto a+c-b \qquad a \longmapsto q-p$$
$$F \longmapsto b+c-a \qquad b \longmapsto q-p$$
$$c \longmapsto q-q=0$$

$$H_0^\Delta(X) = \frac{\ker \partial_0}{\operatorname{im}\partial_1} = \frac{\mathbb{Z}\langle p,q\rangle}{\langle q-p\rangle} \cong \mathbb{Z}$$

$$H_1^\Delta(X) = \frac{\ker \partial_1}{\operatorname{im}\partial_2} = \frac{\mathbb{Z}\langle a-b,c\rangle}{\underbrace{\langle (a-b)+c, (b-a)+c\rangle}} \cong \frac{\mathbb{Z}\langle a-b,c\rangle}{\langle (a-b)+c, 2c\rangle}$$

$$(a-b)+c \;+\; (b-a)+c = 2c$$

$$\cong \frac{\mathbb{Z}\langle \cancel{(a-b)+c}, c\rangle}{\langle \cancel{(a-b)+c}, 2c\rangle} \cong \mathbb{Z}/2\mathbb{Z} \qquad \text{(wow!)}$$

$$H_2^\Delta(X) = \frac{\ker \partial_2}{\operatorname{im}\partial_1} = \frac{0}{0} = 0$$

$\ker \partial_2 = ?$ $\quad \partial E = (a-b)+c \quad \partial F = (b-a)+c$

No linear combo of E and F is in the kernel!

$$H_n^\Delta(X) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2\mathbb{Z} & n=1 \\ 0 & \text{o/w} \end{cases}$$

Rmk. It may be disturbing that $H_2^\Delta(X)=0$, since X is a 2D object! It turns out that the solution is to use different coefficients: $\quad (a-b)+c = (b-a)+c$ over $\mathbb{F}_2$!

(II) Quick Start: classification of surfaces

Munkres Topology

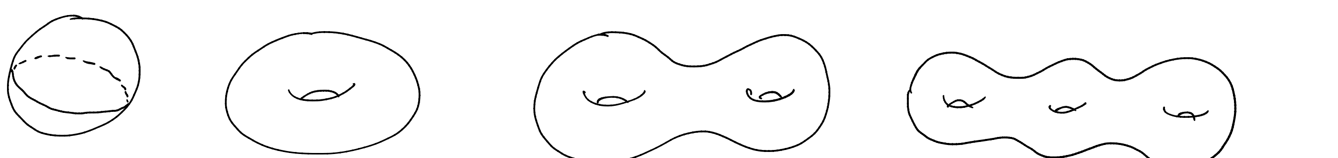
No proof; just know the statement and be able to draw the cell decomposition (ie. polygon depiction of quotient space)

An unfortunately overloaded term:

defn A manifold is <u>closed</u> if it is compact and has empty boundary. (This definition is intrinsic to the space — there is no embedding in an ambient $\mathbb{R}^N$!)

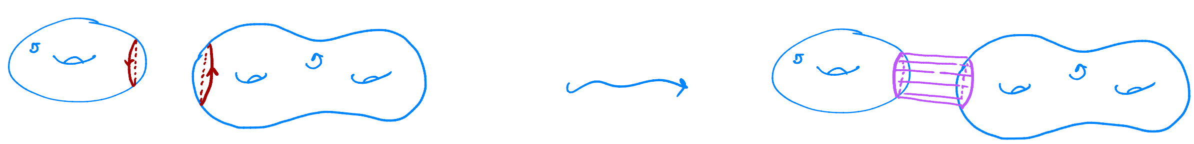
thm. Every closed, connected, orientable ("cco") surface is homeomorphic to one of the spaces in this list:

g = genus   Euler char $= \chi = 2g-2$.



S^2 $T^2 = S^1 \times S^1$ $\Sigma_2 = T^2 \# T^2$ $\Sigma_3 = \#^3 T^2$

<u>Connected sum (#) of two oriented manifolds</u>



Delete nbhds of points   identify boundaries (orientation reversing on the ∂ so that orientations on the two halves are preserved)

<u>Result</u>:

glued!

Construction

For $g \geq 1$, the surface (i.e. 2-manifold) Σ_g is a quotient space of the $4g$-gon with identifications given by the word

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = \prod_{i=1}^{g} \underbrace{[a_i, b_i]}_{\uparrow}$$

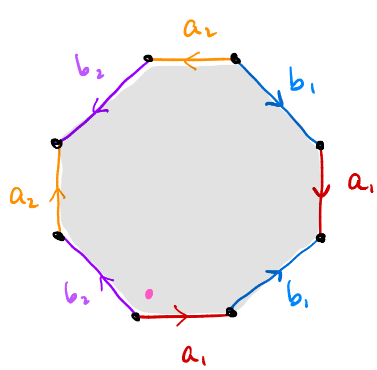
The commutator of a_i and b_i

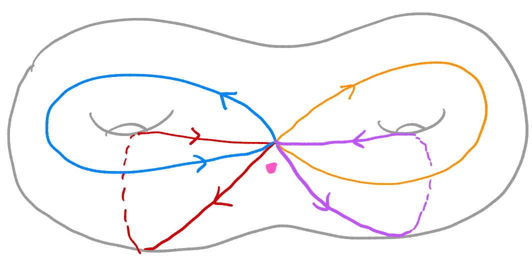
First two examples (pg 5 Hatcher)

$\boxed{g=1}$

Boundary word: $a_1 b_1 a_1^{-1} b_1^{-1}$

$\boxed{g=2}$

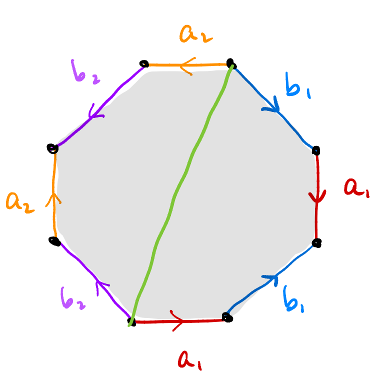


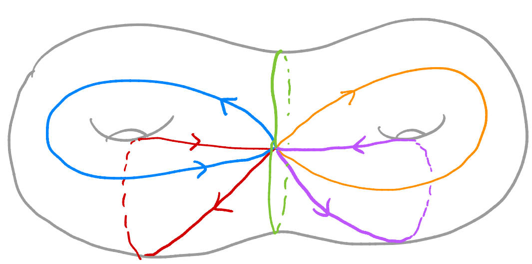


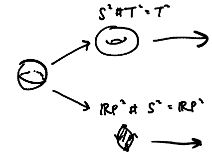
Boundary word: $(a_1 b_1 a_1^{-1} b_1^{-1})(a_2 b_2 a_2^{-1} b_2^{-1})$

Why does this work? Idea:

green = ∂ of connect sum region!







We have a corresponding theorem & construction

<u>thm</u> Every closed, connected, non-orientable surface is homeo to one of the spaces in this list:

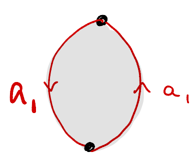
γ = non-orientable genus $\qquad \chi = 2-\gamma \qquad$ (sphere w/ γ "cross-caps")

$\gamma=1$	$\gamma=2$	$\gamma=3$	
$\mathbb{RP}^2$	$\mathbb{RP}^2 \# \mathbb{RP}^2$ (Klein bottle!)	$\#^3 \mathbb{RP}^2$	... and so on

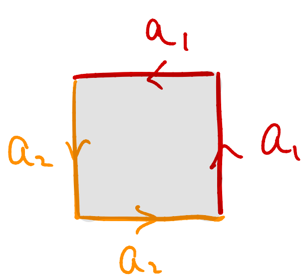
<u>Construction</u>

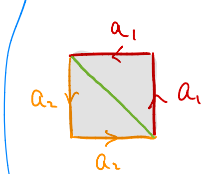
For $\gamma \geq 1$, the non-orientable surface $\#^\gamma \mathbb{RP}^2$ is a quotient space of the 2γ-gon with identifications given by the word $\prod_{i=1}^{\gamma} a_i^2$.

$\mathbb{RP}^2$:



$\mathbb{RP}^2 \# \mathbb{RP}^2$:
Klein bottle!



(


∂ of connect sum regions!
)

III CW-complex (a.k.a. cell complex, cellular complex)

pg. 5 of Hatcher

defn The disk / ball of dim n, denoted D^n or B^n, is

$\{ p \in \mathbb{R}^n \mid |p| \leq 1 \}$ "n-ball"

The sphere of dim n, denoted S^n, is

$\{ p \in \mathbb{R}^{n+1} \mid |p| = 1 \}$ "n-sphere"

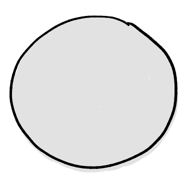
Instead of simplices, we build spaces from cells:

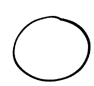
0-cell: $e^0 = \bullet$ a point.

$\partial e^0 = \emptyset$ = "S^{-1}"

1-cell: $e^1 =$  an interval.

$\partial e^1 = \bullet \quad \bullet$ $= S^0$ (two points)

2-cell: $e^2 =$  a disk (D^2)

$\partial e^2 =$  $= S^1$

n-cell $e^n \cong D^n$ $\partial e^n = S^{n-1}$

For $n \geq 1$, ∂e^n is glued onto a complex of smaller dimension. (Less strict gluing rules!)

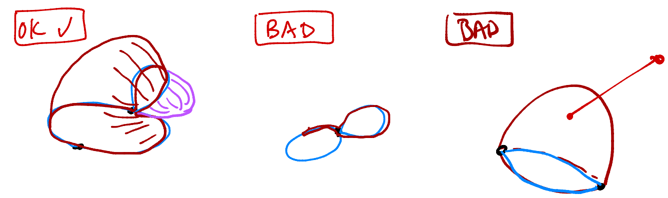
Construction of cell complex:

(1) Start with a discrete set of 0-cells X^0.

(X^0 is the 0-skeleton)

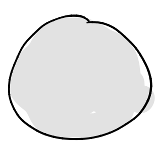
(2) Inductively build the n-skeleton X^n from the $(n-1)$-skeleton X^{n-1} by attaching n-cells e^n_α to X^{n-1}, via maps $\varphi_\alpha : \partial e^n \cong S^{n-1} \longrightarrow X^{n-1}$

(no extra restrictions*!!!)



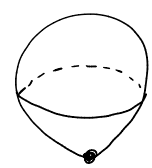
* More details in Hatcher.

eg. we can build S^2 with just two cells:

$\bullet \; \cup$ 

$e^0_\alpha \qquad e^2_\beta$

$\varphi_\beta : \partial e^2_\beta \longrightarrow X^0$ crushes S^1 to a pt.

$S^2 \cong X^2 =$  (balloon).

Rmk. There is CW homology, and honestly its how we usually compute homology of spaces.

Will talk about later, when we have more generality...

We focus so hard on simplicial homology b/c its relationship to singular homology, which is one of the most popular ways homology shows up in proofs...

↳ I'll hopefully give an example to support this later in the course...

IV Remark on ______ complex structures

There are many related terms:

- Δ-complex structure
- cellular structure
- simplicial structure
- triangulation

(see the Wikipedia page, ncatlab page)

<u>defn</u> A (geometric) simplicial complex K is a set of (geometric) simplicies s.t.

(1) Every face of a simplex from K is also in K

(compare w/ Δ-2)

(2) If two simplicies σ_1 and σ_2 intersect nontrivially, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

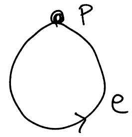
(this is stricter than for Δ-cpxs!)

Also see "abstract simplicial complex" K.

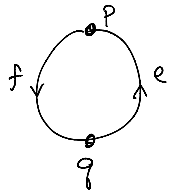
Geometric realization: $|K|$.

<u>Observe</u> that b/c of the constructive nature of this defn, (i.e. X is built from a collection of simplicies), the analogous conditions to Δ-1, Δ-3 need not be listed

↳ Think about this until you agree! It's somewhat subtle....

eg. For S^1 $X =$ 

is a valid simplicial structure

but $Y =$ 

is not!

Problem with Y: $\sigma_e \cap \sigma_f = \begin{matrix} \bullet\ p \\ \bullet\ q \end{matrix}$ (not a simplex!)

defn A triangulation of a space X is an identification of X with a simplicial structure : $|K| \to X$.

Bird's Eye View

stricter ↑

- Smooth structure
- Simplicial structure / triangulation
- CW structure
- (strongly) homotopy equiv to a CW cpx
- topological space

lax-er ↓

Anyways, this is getting too far away from the aims of the course — non-pathological stuff :).

Lecture 6

- Singular homology definition
- ("example") $H_0(X) = \mathbb{Z}^{|\pi_0(X)|}$
- path cpnts decomp.
- point: singular hom better for proofs.
- reduced homology

- algebra: chain maps + induced maps on homology.

Admin

- HW 01 due 9pm
  - submit something
  - typed

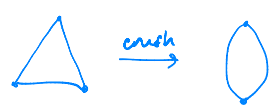
<u>defn</u> Given a topological space X: (very nonrestrictive)

singular chain cpx: $\cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$

- (singular) n-chains :

$$C_n = \mathbb{Z}\langle \{ \text{continuous maps } \sigma_\alpha : \Delta^n \longrightarrow X \} \rangle$$

- that's a lot (uncountable, usually)
- no homeo (on interior) condition! $\Longrightarrow$ "singular"



✓ ok!

- differential ("boundary map") same! on generators:

$$\partial \sigma_\alpha = \sum_{i=0}^{n} (-1)^i \, \sigma_\alpha \big|_{[v_0, \dots, \hat{v_i}, \dots, v_n]}$$

Indeed, restriction is a n-1 chain!

↳ Mind blowing... S(X) is a Δ-cpx...

<u>prop.</u> $\partial^2 = 0$ pf. Same as in simplicial case!

<u>defn.</u> Singular homology of X: $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$

Example? we can do $X = *$ (a point!) ⟶ later today; motivates reduced homology.

Proofs instead of examples!

Hatcher discussion: geometric interpretation ...

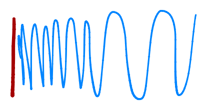
Observation:

<u>Prop. 2.6</u> If $X = \coprod_{\alpha \in A} X_\alpha$ where the X_α are the path components, then $H_n(X) \cong \bigoplus_{\alpha \in A} H_n(X_\alpha)$ for all n.

Pf.

Key: $\operatorname{im} \sigma_\alpha$ always path cntd!

The $C_n(X)$, ∂_n both split as direct sum. (Hence as ker, im, and homology) //

Q. $H_*(X)$ where $X =$  ?

y axis $\cup$ top sine?
b/w $[-1,1]$

<u>Prop 2.7</u> If $X \neq \emptyset$ and is path-cntd, then $H_0(X) \cong \mathbb{Z}$.

$\hookrightarrow$ <u>Cor</u> If $X = \coprod_{\alpha \in A} X_\alpha$ (path cpnts) then $H_0(X) \cong \bigoplus_{\alpha \in A} \mathbb{Z}$

Pf.

$$H_0(X) = C_0(X) / \operatorname{im} \partial_1$$

Def. (an augmentation!) $\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$

$$\sum_i c_i \sigma_i \longmapsto \sum_i c_i$$

— clearly a surjective hom.

<u>Claim</u> If X is path-cntd, $\ker \varepsilon = \operatorname{im} \partial_1$

$\hookrightarrow \Rightarrow \varepsilon : C_0(X) \to \mathbb{Z}$ induces an <u>isom</u>

(Some isom thm...)

$f : M \to N$

$f_* : M / \ker f \xrightarrow{\cong} N$

$$\varepsilon_* : \underbrace{C_0(X) / \ker \varepsilon}_{= C_0(X) / \operatorname{im} \partial_1 = H_0(X)} \longrightarrow \mathbb{Z}$$

ε_* is isom $H_0(X) \cong \mathbb{Z}$.

pf of claim WTS $\ker \varepsilon = \operatorname{im} \partial_1$. Double inclusion!

$\boxed{\operatorname{im} \partial_1 \subset \ker \varepsilon}$ given $\sigma : \Delta^1 \to X$,

$$\varepsilon \partial_1(\sigma) = \varepsilon\left(\sigma|_{[v_1]} - \sigma|_{[v_0]}\right) = 1 - 1 = 0$$

$\boxed{\ker \varepsilon \subset \operatorname{im} \partial_1}$

Suppose $\varepsilon\Big(\underbrace{\sum_i c_i \sigma_i}_{\text{some genuine 0-chain}}\Big) = 0.$, ie. $\sum c_i = 0.$.

Each σ_i is a sing. 0-simplex $\sigma_i : \Delta^0 = pt \longrightarrow X$.

Setup. Choose basepoint $x_0 \in X$.
let σ_0 be the singular 0-simplex w/ image the basepoint x_0.

rmk. basepoints: will let us define reduced hom. "based top spaces"

Since X is path cntd, we can choose a path

$\tau_i : [0,1] \longrightarrow X$ as 1 simplex: $\tau_i : [v_0, v_1] \longrightarrow X$ (Δ^1 ")

from the chosen basepoint x_0 to $\sigma_i(v_0)$.

↳ Observe that τ_i is a sing. 1-simplex!

Then $\partial \tau_i = \sigma_i - \sigma_0$.

$\sum c_i = 0 \Rightarrow$ this is 0!

$$\text{So} \ldots \quad \partial\Big(\underbrace{\sum_i c_i \tau_i}_{\text{a totally legit 1-chain!}}\Big) = \sum_i c_i \sigma_i - \sum c_i \sigma_0 = \sum_i c_i \sigma_i \qquad (!!!)$$

$\Longrightarrow \sum_i c_i \sigma_i$ is a boundary (recall: in $\operatorname{im} \partial_1$).

∎

Reduced homology ~~for based spaces (X, x_0)~~

defn ~~let (X, x_0) be a based space (note $\Rightarrow X \neq \emptyset$)~~

let X be a nonempty space.

The reduced homology of X, $\tilde{H}_*(X)$, is

is the homology of the augmented chain cpx

$$\longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

(where $\tilde{H}_0(X) := \ker(\varepsilon) / \operatorname{im} \partial_1$.)

Rmk. ① X can have multiple path cpnts.

* ② $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$, and $H_n(X) = \tilde{H}_n(X)$ $\forall\, n > 0$.

rmk suspensions + homology gps.

eg. $\mathbb{S}$ spectrum

END