Lecture 7

Goals (this week)

A bunch of algebra

- chain maps, induced maps on homology
- chain htpy, chain htpy equivalence
- exact sequences (short, long)

in support of

- homotopy invariance
- relative homology
- excision

Some students have not seen this

- also how I learned.

sit back + do HW if bored

Chain maps (abs. defn later)

alg

defn Given $\mathcal{C} = (\mathcal{C}, \partial)$ and $\mathcal{C}' = (\mathcal{C}, \partial')$, a chain map $g: \mathcal{C} \to \mathcal{C}$ is a collection of homs $\{g_i : C_i \to C_i'\}$ that commutes with the diff'ls:

$$\begin{tikzcd} C_i \arrow[r, "\partial_i"] \arrow[d, "g_i"'] \arrow[dr, phantom, "\circlearrowright"] & C_{i-1} \arrow[d, "g_{i-1}"] \\ C_i' \arrow[r, "\partial_i'"] & C_{i-1}' \end{tikzcd}$$

"commutative diagram"

"chain condition"

$$\boxed{\partial g = g \partial}$$

top

⚠ Try my best to use this notation!

Given spaces $X, Y \rightsquigarrow$ chain cpxs $C_\bullet(X)$, $C_\bullet(Y)$.

A map $f: X \to Y$ of top spaces (???) (i.e. homeo!)

induces maps on the chain cpxs:

"f-sharp"

$$f_\# \partial(\sigma) = f_\# \Big(\sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

$$= \sum_i (-1)^i f \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} = \partial f_\#(\sigma)$$

idea "induced"

the "obvious" related maps.

$$\begin{tikzcd} \Delta^n \arrow[d, "\sigma"'] \arrow[dr, dashed, "f_\# \sigma"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(composition)

Pictorially at chain level: Keep on board

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+1}(X) \arrow[r, "\partial^X_{i+1}"] \arrow[d, "(f_\#)_{i+1}"] \arrow[dr, phantom, "\circlearrowright"] & C_i(X) \arrow[r, "\partial^X_i"] \arrow[d, "(f_\#)_i"] \arrow[dr, phantom, "\circlearrowright"] & C_{i-1}(X) \arrow[r] \arrow[d, "(f_\#)_{i-1}"] & \cdots \\
\cdots \arrow[r] & C_{i+1}(Y) \arrow[r, "\partial^Y_{i+1}"'] & C_i(Y) \arrow[r, "\partial^Y_i"'] & C_{i-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

core idea
*****.

prop. chain maps induce maps on homology.

pf.

- g takes cycles to cycles:

$$\underbrace{\partial g(\sigma)}_{\text{cycle in } \mathcal{C}'} = g\,\partial(\underset{\substack{\uparrow \\ \text{cycle in } \mathcal{C}}}{\sigma}) = g(0) = 0$$

cycle in $\mathcal{C}'$ $\Leftarrow$ cycle in $\mathcal{C}$

- g takes boundaries to boundaries

$$g(\underbrace{\partial\sigma}_{\partial \text{ in } \mathcal{C}}) = \partial(\underbrace{g\sigma}_{\partial \text{ in } \mathcal{C}'})$$

- $\Rightarrow$ can define $g_*: H_*(\mathcal{C}) \longrightarrow H_*(\mathcal{C}')$ by

$$g_*(\overset{\text{equiv class}}{[\sigma]}) = [g\sigma]$$

$\hookrightarrow$ well-defn b/c if $\sigma, \tau \in [\sigma]$, then

$\tau - \sigma = \partial\beta$ (for some $\beta \in \mathcal{C}$)

and so $g(\tau - \sigma) = g\,\partial\beta$

$\Rightarrow [g\tau] - [g\sigma] = [0] \Rightarrow [g\tau] = [g\sigma]$

$\square$

<u>Notation</u> $f: X \to Y$ mor in Top.

$f_\#: C_\bullet(X) \longrightarrow C_\bullet(Y)$ mor in Ch Cpx

Write

$$f_*: H_\bullet(X) \longrightarrow H_\bullet(Y) \quad \text{mor in } g\text{-}\mathbb{Z}\text{mod}$$

for the induced map in homology.

More observations: (also algebraic result)

Prop. (i) $(fg)_* = f_* g_*$ for $X \xrightarrow{g} Y \xrightarrow{f} Z$

"covariant functor"

(ii) $\mathbb{1}_* = \mathbb{1}$ where $\mathbb{1}$ is identity map

pf sketch.

(i)
$$\begin{tikzcd}
\Delta^n \arrow[d, "\sigma"'] \arrow[dr, "g\sigma"] \arrow[drr, "fg\sigma"] & & \\
X \arrow[r, "g"'] & Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

(ii) clear!

//

Chain homotopy (Write topo. item first - next page)

$$
\begin{tikzcd}
\cdots \arrow[r] & C_{i+1}(X) \arrow[r, "\partial^X"] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] \arrow[dl, dashed] & C_i(X) \arrow[r, "\partial_i^X"] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] \arrow[dl, "h_i"'] & C_{i-1}(X) \arrow[r] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] \arrow[dl, "h_{i-1}"'] & \cdots \arrow[dl, dashed] \\
\cdots \arrow[r] & C_{i+1}(Y) \arrow[r, "\partial_{i+1}^Y"'] & C_i(Y) \arrow[r, "\partial^Y"'] & C_{i-1}(Y) \arrow[r] & \cdots
\end{tikzcd}
$$

<u>defn</u> A <u>chain htpy</u> b/w chain maps $f, g : \mathcal{C} \to \mathcal{C}'$ is a collection of maps

$h = \{ h_i : C_i \to C'_{i+1} \}$ such that

$$f_i - g_i = \partial h + h \partial. \longleftarrow \boxed{\text{chain htpy condition}}$$

<u>prop</u> If f, g are chain htpic (ie $\exists$ htpy h) then $f_* = g_*$ (ie same homomorphism on H_*).

$\hookrightarrow$ save for later. (Maybe HW).

Important, less trivially proven:

<u>thm 2.10</u> Homotopic maps $f, g : X \to Y$ induce the same hom $f_* = g_* : H_\bullet(X) \longrightarrow H_\bullet(Y)$.

<u>Cor 2.11</u> If $f: X \to Y$ is a htpy equivalence

then $f_* : H_\bullet(X) \longrightarrow H_\bullet(Y)$ is an isom of graded $\mathbb{Z}$-mods.

i.e. $f_n : H_n(X) \longrightarrow H_n(Y)$ is an isom $\forall n$.

<u>Cor.</u> If X is contractible, then $\tilde{H}_\bullet(X) = 0$

where $0 \in$ graded $\mathbb{Z}$-mod.

<u>Pf.</u> (sketch – get the main idea + tools in proof)

(Discussion 3)

<u>defn</u> chain htpy equiv:

$\exists\, g : Y \to X$ s.t. $fg \simeq id_Y$, $gf \simeq id_X$

where $\simeq$ means "chain homotopic to"

Discussion 3

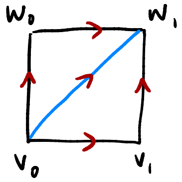
Ⓘ Proof of htpy invariance

Recall Algebraically we want to see a chain htpy $h = \{h_i\}$ as below:

$$\begin{tikzcd}
C_{i+1}(X) \arrow[r] \arrow[d, shift right, "(f_\#)_{i+1}"'] \arrow[d, shift left, "(g_\#)_{i+1}"] & C_i(X) \arrow[r] \arrow[d, shift right, "(f_\#)_i"'] \arrow[d, shift left, "(g_\#)_i"] \arrow[dl, "h_i"'] & C_{i-1}(X) \arrow[d, shift right, "(f_\#)_{i-1}"'] \arrow[d, shift left, "(g_\#)_{i-1}"] \arrow[dl, "h_{i-1}"'] \\
C_{i+1}(Y) \arrow[r] & C_i(Y) \arrow[r] & C_{i-1}(Y)
\end{tikzcd}$$

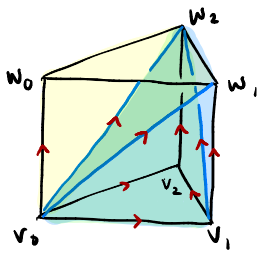
How to: Divide $\Delta^n \times I$ into $(n+1)$-simplicies:

eg. $\boxed{n=1}$



$\boxed{n=2}$

"prism!"



* The ordering is lexicographic with $v < w$, $v_i < v_{i+1}$, $w_i < w_{i+1}$.

To do this carefully, need to talk about barycentric coordinates.

see Hatcher for argument w/ coordinates, more rigorous proof.

- In general, $\Delta^n \times I$ has "bottom" $\Delta^n \times \{0\} = [v_0, \ldots, v_n]$
  "top" $\Delta^n \times \{1\} = [w_0, \ldots, v_n]$.

- We have all the verticies + need to define collection of Δ^{n+1} simplicies. given by the ordered sets

$$[v_0, \ldots, v_i, w_i, \ldots, w_n]$$

if you keep the ordering then

- the chosen $(n+1)$ simplicies won't overlap.
- the result is a simplicial complex.

⚠ Need to check that every point is in a simplex

$\boxed{\text{pf of thm 2.10}}$ $f, g : X \to Y$ htpc.

Given a htpy $F : X \times I \to Y$ from f to g, define (chain homotopies)

<u>prism operators</u> $P : C_n(X) \to C_{n+1}(Y)$ by

$$P(\sigma) = \sum_i (-1)^i \underbrace{F \circ (\sigma \times \mathbb{1}_I)}\Big|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$$

$\sigma : \Delta^n \to X$

$$\Delta^n \times I \xrightarrow{\sigma \times \mathbb{1}} X \times I \xrightarrow{F} Y$$

<u>claim</u> $\partial P = g_\# - f_\# - P\partial$ $\quad (g_\# - f_\# = \partial P + P\partial)$

geom: the ∂ of the big prism

top $\Delta^n \times \{1\}$

bot $\Delta^n \times \{0\}$

sides $\partial \Delta^n \times I$

<u>Pf of claim</u> calculation:

$\partial P(\sigma) =$

$$\sum_i \sum_{j \le i} (-1)^i (-1)^j F \circ (\sigma \times \mathbb{1}_I)\Big|_{[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]}$$

$$+ \sum_i \sum_{j \ge i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \mathbb{1}_I)\Big|_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]}$$

- The case $i=j$ (quick check)
  - $F\circ(\sigma\times\mathbb{1})\big|_{[\hat{v}_0, w_0, \dots, w_n]} = g\circ\sigma = g_\#\sigma$
  - $F\circ(\sigma\times\mathbb{1})\big|_{[v_0, v_1, \dots, v_n, \hat{w}_n]} = -f\circ\sigma = -f_\#\sigma$
  - otherwise they cancel (w/ opp signs)

    eg. $[v_0, \hat{v}_1, w_1, w_2, \dots, w_n] = [v_0, \hat{w}_0, w_1, \dots, w_n]$
- The remaining terms $(i\neq j)$ are exactly $-P\partial(\sigma)$:

  $\sigma\in C_n(X)$

  $$\partial_n\sigma = \sum_j (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_n]}$$

  $$P_{n-1}\circ\partial_n\sigma = \sum_j (-1)^j \underbrace{P_{n-1}\left(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_n]}\right)}$$

  $$= \sum_{i<j} (-1)^i(-1)^j F\circ\underbrace{(\sigma|_{[v_0,\dots,\hat{v}_j,\dots,v_n]}\times\mathbb{1})|_{[v_0,\dots,v_i,w_i,\dots,w_n]}}_{=(\sigma\times\mathbb{1})|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n]}}$$

  $$+ \sum_{i<j} (-1)^i(-1)^{j-1} F\circ\underbrace{(\sigma|_{[v_0,\dots,\hat{v}_j,\dots,v_n]}\times\mathbb{1})|_{[v_0,\dots,v_i,w_i,\dots,w_n]}}_{=(\sigma\times\mathbb{1})|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]}}$$

  * w_j is geometrically at "$v_j\times 1$"

Compare and verify that indeed $-P\partial$ accounts for the remaining terms in ∂P. *geometric interpretation aside

To conclude the proof, prove that if $\exists$ chain htpy h b/w chain maps $f_\#$ and $g_\#$, then $f_* = g_*$. (pure algebra) ☒

Lecture 8

- clarify Cor. 2.11
- $A \subset X$, X/A
- exact seqns, SES, LES

We like sub, quotient objects b/c then we can build more objects from existing ones (eg. modules, spaces...)

<u>Goal</u> Relate $H_\bullet(A)$, $H_\bullet(X)$, and $H_\bullet(X/A)$:

<u>thm 2.13</u>

If X is a space and
A is a nonempty closed subspace that is a deformation retract of some nbhd in X, } <u>good</u> pair (X, A)

then there is an <u>exact sequence</u>

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\delta} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \to \cdots$$

$$\cdots \to \tilde{H}_0(X/A) = 0$$

} LES induced by SES.

a lot of homological algebra; will use regular writing color, except titles.

Exact sequences

↙ works in great generality, eg. Abelian cats.

defn A sequence of morphisms

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

is exact if $\forall n$, $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$.

Note Viewed as chain cpx in particular, this (A, ∂) where $A = \bigoplus A_k$, $\partial = \sum f_k$ is acyclic, ie has trivial homology.

Useful idea for characterizing maps. Eg:

- $0 \to A \xrightarrow{\alpha} B$ is exact iff α is injective
- $A \xrightarrow{\alpha} B \to 0$ ——— iff α is surj.

defn Short exact sequence (SES):

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \longrightarrow 0$$

suggestively: $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are going to be chain cpxs; α, β chain maps.

note α is inj, β is surjective

Long Exact sequences (LES) from SESs of chain complexes

$$
\begin{tikzcd}
0 \arrow[d] \\
\mathcal{A} \arrow[d, "i"'] \\
\mathcal{B} \arrow[d, "p"'] \\
\mathcal{C} \arrow[d] \\
0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial^A"] \arrow[d, "i"'] & A_n \arrow[r, "\partial^A"] \arrow[d, "i"'] & A_{n-1} \arrow[r] \arrow[d, "i"'] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial^B"] \arrow[d, "p"'] & B_n \arrow[r, "\partial^B"] \arrow[d, "p"'] & B_{n-1} \arrow[r] \arrow[d, "p"'] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C"] \arrow[d] & C_n \arrow[r, "\partial^C"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Keep on Board

SES $\longrightarrow$ $\mathbb{Z}$-many SES,

of chain cpxs $\longrightarrow$ plus the maps i, p commute w/ the differentials $\partial^A, \partial^B, \partial^C$.

- These i, p induce maps on homology, called i_*, p_*.

Picture

$$
\begin{tikzcd}
H_\bullet(\mathcal{A}) \arrow[rr, "i_*"] & & H_\bullet(\mathcal{B}) \arrow[dl, "p_*"] \\
 & H_\bullet(\mathcal{C}) \arrow[ul, dashed, "\text{``}\partial\text{''}", "{[1]}"'] &
\end{tikzcd}
$$

- We can also define (and define it well!) a <u>connecting homomorphism</u> $\partial : H_\bullet(\mathcal{C}) \longrightarrow H_\bullet(\mathcal{A})[1]$ homgr preserving

$$\partial = \{ \partial_n : H_n(\mathcal{C}) \longrightarrow H_{n-1}(\mathcal{A}) \}$$

* very unfortunate clashing of notation with diff'l of $\mathcal{A}, \mathcal{B}, \mathcal{C}$; hence carefully wrote $\partial^A, \partial^B, \partial^C$

* This is indeed standard notation, so I will not change it.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\arrow[r] & A_n \arrow[r, "\partial^A"] \arrow[d, "i"'] & A_{n-1} \arrow[r] \arrow[d, "i"'] & {} \\
\arrow[r] & B_n \arrow[r, "\partial^B"] \arrow[d, "p"'] & B_{n-1} \arrow[r] \arrow[d, "p"'] & {} \\
\arrow[r] & C_n \arrow[r, "\partial^C"] \arrow[d] \arrow[uur, bend left] & C_{n-1} \arrow[r] \arrow[d] & {} \\
 & 0 & 0 &
\end{tikzcd}$$

Diagram Chasing!

We wish to define $\partial[c] = [a]$ where a is chosen as below:

$$\begin{tikzcd}
A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[d] \\
B_n \arrow[r] \arrow[d] & B_{n-1} \arrow[d] \\
C_n \arrow[r] & C_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & a \arrow[d, mapsto, "\boxed{3}"'] \\
\exists b \arrow[r, mapsto, "\boxed{2}"] \arrow[d, mapsto, "\boxed{1}"'] & \partial b \\
c &
\end{tikzcd}$$

Things to prove:

I. Given $c \in C_n$, find some a

II. show different choices in the defn of a lead to a' in the same homology class

III. show that different choices of $c \in [c]$ yield the same $[a]$.

IV. ∂ is a homomorphism (easy)

I. Let $c \in [c]$. * note c is a cycle

$$\begin{tikzcd}
A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[d] \\
B_n \arrow[r] \arrow[d] & B_{n-1} \arrow[d] \\
C_n \arrow[r] & C_{n-1}
\end{tikzcd}$$

$\boxed{1}$ $\exists b \mapsto c$

$\boxed{2}$ $\exists b \mapsto \partial b$

$\boxed{3}$ $a \mapsto \partial b$

$\boxed{1}$ p is surjective

$\boxed{2}$ use ∂ to map forward

$\boxed{3}$ • c is a cycle $\Rightarrow \partial^{\mathcal{C}} c = 0$

• Bottom square commutes $\Rightarrow p(\partial b) = 0$.

ie. $\partial b \in \ker p$.

• Right column is exact $\Rightarrow \partial b \in \operatorname{im} i$

$\Longrightarrow \exists\, a$ s.t. $i(a) = \partial b$.

II. the choices made in I do not matter:

$\boxed{1}$ i is injective so there was no choice made

$\boxed{2}$ made no choices

$\boxed{3}$ Suppose we chose a different lift $b' \in p^{-1}(c)$.

Then $b - b' = i(a')$ for some $a' \in A_n$ (since $\ker p = \operatorname{im} i$).

Then $\partial b' = \partial\big(b - i(a)\big) = \partial b + \partial i(a')$.

Since i is a chain map, $\partial i(a') = i\,\partial a'$

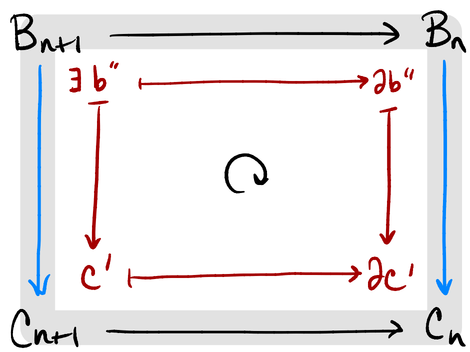
$\Longrightarrow$ we obtain $a + \partial a' \in [a]$.

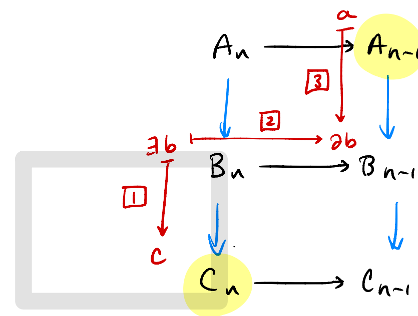
III. Our choice of $c \in [c]$ did not matter either.

Any other choice of representative would be of the form

$c + \partial c'$ where $c' \in C_{n+1}$.

$\boxed{1}$





May choose $b + \partial b'' \in B_n$.

Then $\partial(b + \partial b'') = \partial b + 0$.

Done! Choice $c + \partial c' \in [c]$ did not matter!

IV. ∂ is a homomorphism (easy)

left to reader :)

Lecture 9

- still need to teach applications @ end of class
- first, the hard proof

Main thm for today:

<u>thm 2.16</u> The sequence of homology groups

$$\cdots \to H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \cdots$$

is exact.

pf.

<u>Q</u>. What do we need to check?

im $\subset$ ker statements:

① $j_* i_* = 0$

② $\partial j_* = 0$

③ $i_* \partial = 0$

ker $\subset$ im statements

④ $\ker j_* \subset \operatorname{im} i_*$

⑤ $\ker \partial \subset \operatorname{im} j_*$

⑥ $\ker i_* \subset \operatorname{im} \partial$

① $j \circ i = 0 \implies j_* \circ i_* = 0$

homology is a (covariant functor)

② $\partial j_* = 0$

If $[b] \in H_n(B)$ then b is a cycle $\Rightarrow \partial^B(b) = 0$

$$
\begin{tikzcd}
A_n \arrow[d] & & A_{n-1} \arrow[d] \\
B_n \arrow[d] \arrow[rr, "\partial^B"] & & B_{n-1} \arrow[d] \\
C_n & & C_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 0 \arrow[d, mapsto] \\
b \arrow[d, mapsto] \arrow[r, mapsto] & 0 \\
c &
\end{tikzcd}
$$

$\Rightarrow \partial[\underset{j_*[b]}{\overset{\shortparallel}{c}}] = 0.$

③ $i_* \partial = 0$

$$
\begin{tikzcd}
A_n \arrow[d] & & A_{n-1} \arrow[d] \\
B_n \arrow[d] \arrow[rr, "\partial^B"] & & B_{n-1} \arrow[d] \\
C_n & & C_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & a \arrow[d, mapsto] \\
b \arrow[d, mapsto] \arrow[r, mapsto] & \partial^B b \\
c &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & {[a]} \arrow[d, "i_*"] \\
{[c]} \arrow[ur, "\partial"] & {[\partial^B b] = 0}
\end{tikzcd}
$$

④ $\ker j_* \subset \operatorname{im} i_*$

Let $[b] \in \operatorname{Ker} j_*$

$\Rightarrow [jb] = 0$

$\Rightarrow jb = \partial c'$ for some $c' \in C_{n+1}$

j is surjective $\Rightarrow c' = j(b')$ f.s. $b' \in B_{n+1}$.

Now.... $j(b - \partial b') = j(b) - j(\partial b') \overset{\text{ch map}}{=} j(b) - \partial j(b') = 0$

b/c $\quad \partial j(b') = \partial c' = j(b)$.

$\Rightarrow b - \partial b' \in \ker j = \operatorname{im} i \Rightarrow b - \partial b' = i(a)$ f.s. $a \in A_n$.

a is a cycle:

$i(\partial a) = \partial i(a) = \partial(b - \partial b') = \partial b = 0$

and i is injective.

So $\quad i_*[a] = [b - \partial b'] = [b] - [\partial b'] = [b]$.

$\Rightarrow$ we have found $[a] \in H_n(A)$ that maps onto arbitrary $[b] \in \ker j_*$.

⑤ $\ker \partial \subset \operatorname{im} j_*$

⑥ $\ker i_* \subset \operatorname{im} \partial$

Similar flavor, not instructive for me to walk through in class

* highly recommended that you work through these on your own for your own notes / practice.

Relative Homology Groups

Obs. We can view
$$C_n(A) \hookrightarrow C_n(X)$$
$$\sigma_\alpha \longmapsto \tau_\alpha$$

$$\begin{tikzcd} \Delta^n \arrow[d, "\sigma_\alpha"'] \arrow[dr, dashed, "\tau_\alpha = i\sigma_\alpha"] & \\ A \arrow[r, hook, "i"'] & X \end{tikzcd}$$

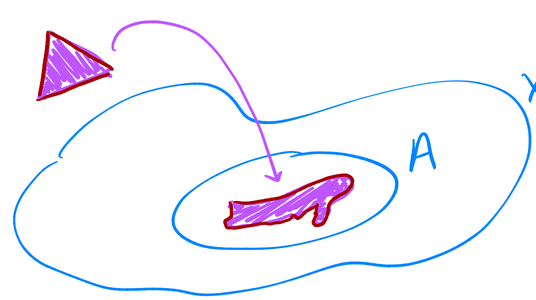
Injective: If $i\sigma_\alpha = i\sigma_\beta$, then $\sigma_\alpha = \sigma_\beta$.

defn/thm (Relative Homology)

- Given $A \subset X$, let $C_n(X, A) := C(X)/C(A)$

- The bd. map $\partial_x : C_n(X) \longrightarrow C_{n-1}(X)$ induces
$$\partial : C_n(X, A) \longrightarrow C_{n-1}(X, A)$$
since $\partial_x\big(C_n(A)\big) \subseteq C_{n-1}(A)$

think about it...



So we may define the relative singular chain cpx:

$$\cdots \to C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \xrightarrow{\partial} \cdots$$

- $\partial^2 = 0$ since $\partial_X^2 = 0$ already.

- relative cycles:

  n-chains $\tau \in C_n(X)$ s.t. $\partial\tau \in C_{n-1}(A)$

- relative boundaries:

  $$\tau = \partial\tau_1 + \tau_2 \text{ where } \tau_1 \in C_{n+1}(X), \quad \tau_2 \in C_n(A)$$

$\rightsquigarrow$ $H_n(X,A)$ really is homology of X mod (homo of) A.

<u>Thm 2.13</u> If (X,A) is a good pair,

X is a space and A is a nonempty closed subspace that is a deformation retract of some nbhd in X,

then there is an exact sequence

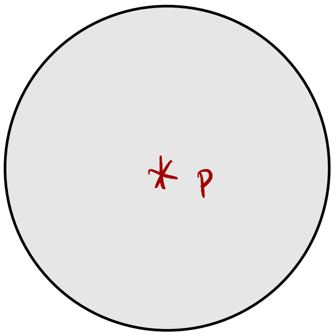
$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\delta} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \to \cdots$$

$$\cdots \to \tilde{H}_0(X/A) = 0$$

LES induced by SES.

<u>Rmk</u> Where do we use good pair? We'll see Excision Thm next, whose pf is more geometric. Then we'll see that when (X,A) is good, $H_\bullet(X/A) = H_\bullet(X,A)$.

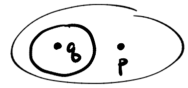
eg. Not good pair: $(A \subset X$ open$)$



$A = \mathbb{D} - \{p\} =: \mathbb{A}$ "annulus"

$X = \mathbb{D}$

$X/A \cong \{q, p\}$ with topology



"Sierpiński space"

(non Hausdorff!)

algebra: $H_n(\mathbb{D}, \mathbb{A})$

topo: $H_n(\mathbb{D}/\mathbb{A})$

discrepancy when (X, A) is not good.

$H_\cdot(X/A) \neq H_\cdot(X, A)$ in this case!

If we applied the LES we would get

$$\underset{0}{\tilde{H}_2(\mathbb{A})} \longrightarrow \underset{0}{\tilde{H}_2(\mathbb{D})} \longrightarrow \underset{\Rightarrow \mathbb{Z} \ \ldots}{\tilde{H}_2(\mathbb{D}, \mathbb{A})}$$

$$\hookrightarrow \underset{\mathbb{Z}}{\tilde{H}_1(\mathbb{A})} \longrightarrow \underset{0}{\tilde{H}_1(\mathbb{D})} \longrightarrow \underset{\Rightarrow 0}{\tilde{H}_1(\mathbb{D}, \mathbb{A})}$$

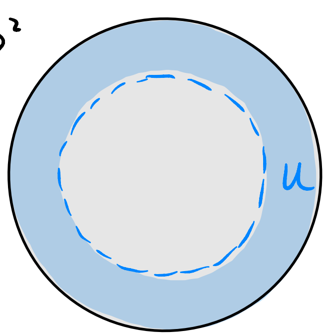
$$\hookrightarrow \underset{0}{\tilde{H}_0(\mathbb{A})} \longrightarrow \underset{0}{\tilde{H}_0(\mathbb{D})} \longrightarrow \underset{\Rightarrow 0}{\tilde{H}_0(\mathbb{D}, \mathbb{A})} \longrightarrow 0$$

$\Rightarrow 0 \rightsquigarrow$ actually $\exists$ path $p \rightsquigarrow q$ ✓.

eg. good pair $S^1 \hookrightarrow \mathbb{D}^2 \longrightarrow S^2$

The calculation above is now valid.

$\mathbb{D}^2$



$S^1 = \partial \mathbb{D}^2$ is def retract of open nbhd $U \subset \mathbb{D}^2$.

* some app to fixed pts skipped Cor. 2.15.