Lecture 10

Last week: $A \subset X$, relate $H_\bullet(A)$ and $H_\bullet(X)$ via LES w/ $H_\bullet(X,A)$

We still have htpy invariance even for $H_\bullet(X,A)$:

<u>prop 2.19</u> If two maps $f, g : (X,A) \longrightarrow (Y,B)$ are htpc through maps of pairs $(X,A) \longrightarrow (Y,B)$, then $f_* = g_* : H_\bullet(X,A) \longrightarrow H_\bullet(Y,B)$

Pf. Check that the prism operator induces relative prism operator

$$P_n : C_n(X,A) \longrightarrow C_{n+1}(Y,B)$$

(check $C_n(A) \rightsquigarrow C_{n+1}(B)$ under P.)

Then in the quotient, the htpy condition $\partial P + P\partial = g_\# - f_\#$ still holds //

Later, we'll try to state most things in terms of relative homology $H_\bullet(X,A)$ for more generality.

Generalization: LES of a triple of spaces

<u>defn</u> Triple: (X, A, B) where $B \subset A \subset X$

<u>thm.</u>

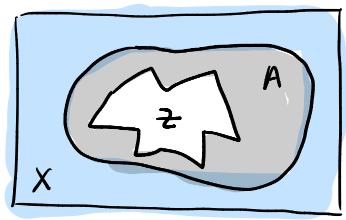
$$\begin{tikzcd}
H_n(A,B) \arrow[rr] & & H_n(X,B) \arrow[dl] \\
& H_n(X,A) \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

from the SES (of quotient cpxs)

$$0 \to C_\bullet(A,B) \longrightarrow C_\bullet(X,B) \longrightarrow C_\bullet(X,A) \longrightarrow 0$$

[? when useful.? Wait a bit; will use in conjunction w/ excision to show thm 2.13 finally ($\tilde{H}(X,A) \cong \tilde{H}(X/A)$ for good pair).]

Excision



$Z \subset A \subset X$ (allow generality)

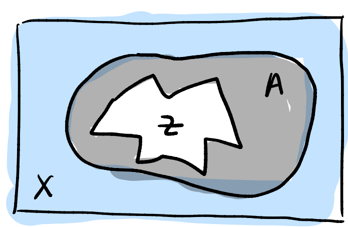
Want to excise (cut out) Z.

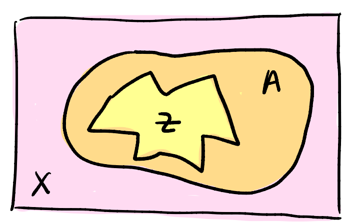
<u>thm 2.20</u> (Excision thm)

> Given $Z \subset A \subset X$ where $\overline{Z} \subset \mathring{A}$, then $(X-Z, A-Z) \hookrightarrow (X, A)$ induces isoms
>
> $$H_n(X-Z, A-Z) \xrightarrow{\cong} H_n(X, A) \quad \forall n.$$

graded statement that I will write henceforth:

$$H_\bullet(X-Z, A-Z) \xrightarrow{\cong} H_\bullet(X, A)$$





$\mathring{B} = X - \overline{Z} =$

$A =$

(color theory)

> Equivalently, for subspaces $A, B \subset X$ s.t.
>
> $\mathring{A} \cup \mathring{B} = X$ interiors cover X
>
> the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isoms
>
> $$H_\bullet(B, A \cap B) \longrightarrow H_\bullet(X, A).$$

We'll do proof of excision tomorrow or wednesday.

More importantly, need to see how its useful.

proof of thm 2.13 now only requires

If (X,A) is a good pair, then $H_\bullet(X,A) \cong H_\bullet(X/A)$.

prop. 2.22 For good pairs (X,A), the quotient map

$$q : (X,A) \longrightarrow (X/A, A/A)$$

induces isomorphisms

$$q_* : H_\bullet(X,A) \longrightarrow \underbrace{H_\bullet(X/A, A/A) \approx \tilde{H}_n(X/A)}$$

on HW; this is how we go from relative to absolute homology groups

pf.

Let V be a nbhd of A in X that def. retracts onto A.

(Maps below are the obvious induced ones from inclusion)

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "① \cong"] \arrow[d, "q_*", "⑥ \cong"'] & H_n(X,V) \arrow[d, "q_*"] & H_n(X-A, V-A) \arrow[l, "③ \cong"'] \arrow[d, "\cong q_*", "⑤"'] \\
H_n(X/A, A/A) \arrow[r, "② \cong"] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "\cong ④"']
\end{tikzcd}$$

Goal

Show ① - ⑤ are isos. Since we have big square commutes, we'll get ⑥ is isom, which is what we want.

① & ②:

- $\tilde{H}_n(X,A) \xrightarrow{\iota_*} \tilde{H}_n(X,V)$ shows up in LES of triple buttressed by $\tilde{H}_n(V,A)$ and $\tilde{H}_{n-1}(V,A)$.

- The def. retract $V \rightsquigarrow A$ induces def. retract of pairs $(V,A) \rightsquigarrow (A,A)$.

  - By homotopy equivalence of homology of a pair (prop. 2.19)

  $H_\bullet(V,A) \cong H_\bullet(A,A)$ (reduced or not)

  - But the chains $C_\bullet(A,A)$ are all 0.

  $\Rightarrow H_\bullet(A,A) = 0. \Rightarrow H_\bullet(V,A) = 0$

- So $\tilde{H}_n(X,A) \xrightarrow{\iota_*} \tilde{H}_n(X,V)$ is an isom

③ & ④ These are $\cong$s by the Excision theorem!

⑤ At the chain level, the complexes can be identified because $q|_{X-A}$ is a homeo on these spaces!

$\Rightarrow q_*$ is an isom on homology

$\Rightarrow$ since the squares commute, ⑥ must also be an isom. ■

$\mathcal{H}$

Intro to idea of barycentric subdivision

We need to do this for

1. geometric simplices
2. linear chains (singular simplices)

cone operator...

Do this in paper.

Discussion 4 or lecture 11

Equivalently, for subspaces $A, B \subset X$ s.t.

$$\mathring{A} \cup \mathring{B} = X \qquad \text{interiors cover } X$$

the inclusion $(B, A\cap B) \hookrightarrow (X, A)$ induces isoms

$$H_\bullet(B, A\cap B) \longrightarrow H_\bullet(X, A).$$

Space X.

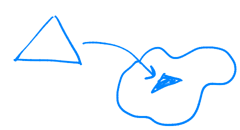
$\mathcal{U} = \{U_j\}$ coll of spaces s.t. $\{\mathring{U}_j\}$ is an open cover of X.

- Let $C_n^{\mathcal{U}}(X) \subset C_n(X)$ be the subgroup gen'd by

  σ_α where $\operatorname{im}(\sigma_\alpha) \subset U_{j_\alpha}$ for some j_α.

  (σ_α lands in one of the sets)

- (clearly) $\partial : C_n(X) \longrightarrow C_{n-1}(X)$ takes

  $$C_n^{\mathcal{U}}(X) \longrightarrow C_{n-1}^{\mathcal{U}}(X).$$

  $\Rightarrow \partial^2 = 0$, and the $C_\bullet^{\mathcal{U}}(X)$ form a chain cpx.

  $\rightsquigarrow H_n^{\mathcal{U}}(X)$

- WTS $H_n^{\mathcal{U}}(X) \cong H_n(X)$

chain htpy equivalence, $\Rightarrow$ isom on homology.

2

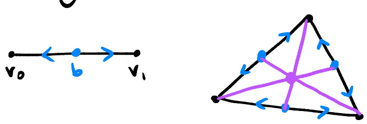
prop. 2.21 The inclusion $\iota : C_n^{\mathcal{U}}(X) \longrightarrow C_n(X)$ is a chain htpy equivalence, i.e. $\exists\ \rho : C_n(X) \longrightarrow C_n^{\mathcal{U}}(X)$

s.t. $\iota\rho \simeq \mathbb{1}_{C_\bullet(X)}$, $\rho\iota \simeq \mathbb{1}_{C_\bullet^{\mathcal{U}}(X)}$

$\Longrightarrow$ ι induces isoms $H_n^{\mathcal{U}}(X) \cong H_\bullet(X)$.

pf:

(1) Barycentric subdivision



Fact diam of each resulting "smaller simplex" is

$\leq \frac{n}{n+1}$ diam Δ^n

* size + simplicial comment

$\Rightarrow$ with r applications we can get simplicies $\left(\frac{n}{n+1}\right)^r$ the original diam.

$\frac{n}{n+1} < 1 \Rightarrow \lim \left(\frac{n}{n+1}\right)^r = 0$:)

Want to construct a subdivision operator

$S: C_n(X) \longrightarrow C_n(X)$, and show $S \simeq \mathbb{1}_{C.(X)}$

Then $S_* = \mathbb{1}_*$, ie subdividing doesn't change the homology.

(2) B. Sub. of linear chains

simpler setting

(only summary - used paper notes)

Consider $Y \subset \mathbb{R}^N$, convex.

- $LC_n(Y) \subset C_n(Y)$ is gen'd by linear maps $\Delta^n \longrightarrow Y$.

  linear chains

- Work inductively using augmented complex, $\widetilde{LC}.(Y)$
$\widetilde{LC}_{-1}(Y) = \mathbb{Z}\langle[\emptyset]\rangle$, where for any 0-simplex $[w_0]$, $\partial[w_0] = [\emptyset]$.

- Identify linear chain $\lambda: \Delta^n \longrightarrow Y$ w/ $[w_0, \dots, w_n]$ where $w_i = \lambda(v_i)$.

- Define hom $b: LC_n(Y) \longrightarrow LC_{n+1}(Y)$ for $b \in Y$.
$[w_0, \dots, w_n] \longmapsto [b, w_0, \dots, w_n]$

$\rightsquigarrow$ $\partial b + b\partial = \mathbb{1}$ (-0).

- Use b_λ = barycenter of $[w_0, \dots, w_n]$ to define inductively subdivision operator $S(\lambda) = b_\lambda(S\partial\lambda)$ w/ $S_{-1}, S_0 = \mathbb{1}$.

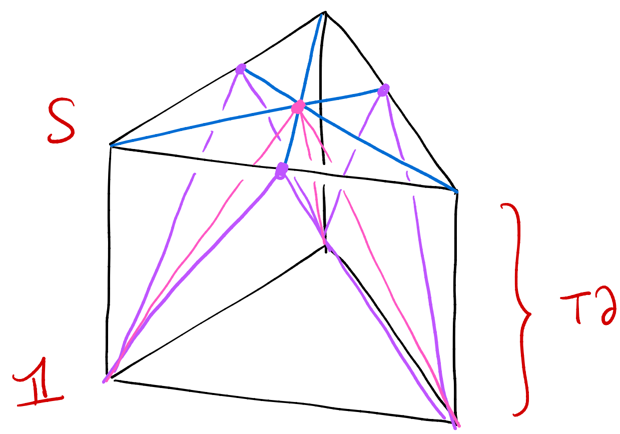
  - show S is a chain map.

- Then define a chain htpy T b/w S and $\mathbb{1}$: Inductively

$$T_{-1} = 0 \qquad T\lambda = b_\lambda(\lambda - T\partial\lambda)$$

- show this is a chain htpy $T\partial + \partial T = \mathbb{1} - S$

∂T = "boundary of T"



- Having built these maps, revert back to $LC_\bullet(Y)$ (from $\widetilde{LC}_\bullet$)
  - $\partial T + T\partial = \mathbb{1} - S$ still holds since $T_{-1} = 0$.

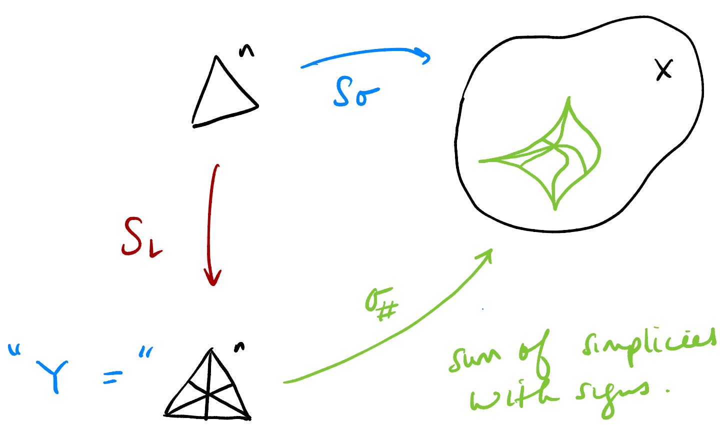
(3) Bary. subdiv of general chains

<u>defn</u> $S : C_n(X) \longrightarrow C_n(X)$

from S_L = the subdiv op. from last time as

$$S\sigma = \sigma_\# S_L \Delta^n$$

view $\sigma_\#$ as chain map induced by $\Delta^n \xrightarrow{\sigma} X$



$$\begin{tikzcd}
LC_n(\Delta^n) \arrow[d, "S_L"'] \arrow[drr, "S\sigma"] & & \\
LC_n(Y) \arrow[r, hook] & C_n(Y) \arrow[r, "\sigma_\#"] & C_n(X)
\end{tikzcd}$$

WTS S is a chain map: WTS $\partial S = S\partial$ (Conceptually not complicated) ④

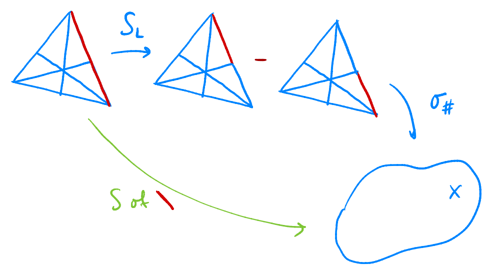
$$\partial S\sigma = \partial \sigma_\# S_L \Delta^n = \sigma_\# \partial S_L \Delta^n = \sigma_\# S_L \partial \Delta^n$$

$\sigma_\#$ is chain map $\quad$ S_L is chain map

$$= \sigma_\# S_L \left(\sum_i (-1)^i \Delta_i^n \right)$$

Here Δ_i^n is the ith facet (codim 1 face)

$$= \sum_i (-1)^i \sigma_\# S_L \Delta_i^n$$



$$= \sum_i (-1)^i S\left(\sigma|_{\Delta_i^n}\right)$$

$$= S\left(\sum_i (-1)^i \sigma|_{\Delta_i^n} \right) = S\partial\sigma$$

Similarly define T from T_L:

$$T: C_n(X) \longrightarrow C_{n+1}(X) \qquad T\sigma = \sigma_\# T\Delta^n$$

Check $\partial T + T\partial = \mathbb{1} - 0$ still holds.

Same deal / conceptual picture.

(5)

(4) Iterated.

(what if you need your simplices really small?)

Just need a chain htpy b/w $\mathbb{1}$ and S^m.

<u>defn.</u> operator $D_m = \sum_{0 \le i < m} TS^i$

<u>Check</u>: $\partial D_m + D_m \partial$

$$= \sum_{i=0}^{m-1} \partial T S^i + T S^i \partial$$

S^i is chain map

$$= \sum \underline{\partial T} S^i + \underline{T \partial} S^i$$

T is already htpy

$$= \sum (\mathbb{1} - S) S^i$$

$$= \sum_{i=0}^{m-1} S^i - S^{i+1}$$

<u>telescope</u>!

$$= \mathbb{1} - S^m.$$

6

Profit! (with some more work)

Recall $C_\bullet^{\mathcal{U}}(X)$ .... did we do this yesterday?

From analysis:

Open cover $\mathcal{U}$ of Δ^n, a compact metric space.

$\exists$ $\varepsilon > 0$ (Lebesgue #) s.t. every set V of diam $< \varepsilon$ lies inside some $U \in \mathcal{U}$.

$\Rightarrow$ for each Δ^n we have some $\varepsilon(\Delta^n)$.

For each $\sigma: \Delta^n \to X$, defn. $m(\sigma)$ = smallest m s.t. $S^m \sigma$ is in $C_n^{\mathcal{U}}(X)$. $\leftarrow$ recall this?

We can now define $D: C_n(X) \to C_{n+1}(X)$ by $D\sigma = D_{m(\sigma)}\sigma$ for each simplex σ.

(P)

Goal: Chain map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$

that is a chain htpy inverse for $\iota : C_n^{\mathcal{U}}(X) \to C_n(X)$.

Once we have this, we have chain htpy equiv b/w

$C_\bullet^{\mathcal{U}}(X)$ and $C_\bullet(X)$. (recall prop 2.21)

For each σ, we have $D\sigma = D_{m(\sigma)}\sigma$.

Let $\rho = \mathbb{1} - \partial D - D\partial$ Check chain map ✓.

Then $\rho(\sigma) = \sigma - \partial D\sigma - D\partial\sigma$

$$= \underbrace{\underbrace{\sigma - \partial D_{m(\sigma)}\sigma}_{\mathbb{1}\sigma - \partial D\sigma}}_{S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma)} - D\partial\sigma$$

linear combo of different $D_{m(\sigma')}$

$\partial D_m + D_m\partial = \mathbb{1} - S^m$

$\Rightarrow \mathbb{1} - \partial D = S^m - D\partial$

$$= \underbrace{S^m\sigma}_{\text{in } C_n^{\mathcal{U}}(X)} + \underbrace{D_{m(\sigma)}\partial\sigma - D\partial\sigma}_{\text{linear combos of terms}}$$

in $C_n^{\mathcal{U}}(X)$

linear combos of terms

$D_{m(\sigma)}\sigma_j - D_{m(\sigma_j)}\sigma_j$

where $m(\sigma_j) \leq m(\sigma)$

smaller facets! bigger simplex!

$\Rightarrow$ all terms live in $C_n^{\mathcal{U}}(X)$!

So ρ can be viewed as a map $C_\bullet(X) \to C_\bullet^{\mathcal{U}}(X)$

(previously ρ was $\iota\rho$.)

Claim ρ, ι are htpy inverses

pf

- Now $\rho = \mathbb{1} - \partial D - D\partial \iff \partial D + D\partial = \mathbb{1} - \rho$

$$\Rightarrow \partial D + D\partial = \mathbb{1} - \iota\rho$$

$$C^u_\bullet(X) \hookrightarrow C_\bullet(X) \qquad C_\bullet(X) \longrightarrow C^u_\bullet(X)$$

- $D = 0$ (no htpy needed) on $C^u_\bullet(X)$
$m(\sigma) = 0$

$$\Rightarrow \rho\iota = \mathbb{1}$$

$\left(\rho = \mathbb{1} - \partial D - D\partial \text{ by defn} \right)$

//

end pf of prop. 2.21!

Pf. of excision thm (restatement) on paper.

9

Cover: $X = A \cup B$. Let's just say $A = \mathring{A}$, $B = \mathring{B}$...

Write $C_n(A+B) := C_n^{\mathcal{U}}(X)$ more intuitive notation

The htpy equations $\partial D + D\partial = \mathbb{1} - \iota\rho$, $\rho\iota = \mathbb{1}$

all descend to the quotient by $C_\bullet(A)$. (check mentally)

$\Rightarrow$ ① $C_n(A+B)/C_n(A) \xrightarrow{\text{"}\bar{\iota}\text{"}} C_n(X)/C_n(A)$

Induces isom on homology

② $C_n(B)/C_n(A\cap B) \xrightarrow{i} C_n(A+B)/C_n(A)$

can be an identification at chain level

$\Rightarrow$ together we have maps inducing isom on homologies:

$$H_n(B, A\cap B) \cong H_n(X, A).$$

Go home and take a deep breath.

Lecture 12

* HW # ? There is a wedgesum corollary in the book, also part (a)

* Midterm next week when I travel
  - accommodations
  - know the results, defns, methods of calculation
    - rigorous exercises will not be long diagram chases! (I will take the exam ahead)

<u>Today</u>
- note on naturality
- sketch of equiv of H^Δ and H: uses 5-lemma

LES odds + ends

① In the LES of a pair, $\partial[\alpha] = [\partial\alpha]$:

$$\begin{tikzcd}
C_n(A) \arrow[d, hook] & & \partial\alpha \arrow[d, mapsto] \\
C_n(X) \arrow[d] & \alpha \arrow[r, mapsto] & \partial\alpha \\
C_n(X,A) & \alpha \arrow[u, mapsto, leftarrow] &
\end{tikzcd}$$

rel. cycle $\Rightarrow \partial\alpha \in C_{n-1}(A)$

② There are two versions: unreduced and reduced

thm 2.16 (algebra)

following, we have

- LES for unreduced hom of pair

$$\cdots \to H_0(X,A) \to 0$$

- LES for reduced hom when $A \neq \emptyset$.

Have SES $0 \to \underset{C_{-1}(A)}{\mathbb{Z}} \xrightarrow{id} \underset{C_{-1}(X)}{\mathbb{Z}} \to \underset{C_{-1}(X,A)}{0} \to 0$

in dimension -1.

$$\to \tilde{H}_1(X,A) \to \tilde{H}_0(A) \to \tilde{H}_0(X) \to \tilde{H}_0(X,A) \to \mathbb{Z} \xrightarrow{id} \mathbb{Z} \to 0 \to 0.$$

$$\Rightarrow H_n(X,A) = \tilde{H}_n(X,A) \quad \forall n.$$

Note on Naturality

We have a construction

$$\text{Top pairs} \xrightarrow{\mathcal{G}} \text{SES of chain cpxs} \xrightarrow{\mathcal{F}} \text{LES on homology}$$

(X, A) where $A \subset X$

$$\underbrace{0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0 \qquad \begin{tikzcd} H_\bullet(\mathcal{A}) \arrow[rr] & & H_\bullet(\mathcal{B}) \arrow[dl] \\ & H_\bullet(\mathcal{C}) \arrow[ul, "{[1]}"] & \end{tikzcd}}_{*}$$

$\mathcal{A} = C_\bullet(A)$

$\mathcal{B} = C_\bullet(X)$

$\mathcal{C} = C_\bullet(X, A)$

defn/claim $\mathcal{F}$ is natural:

whenever we have a chain map

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r, "i"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{B} \arrow[r, "p"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C} \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & \mathcal{A}' \arrow[r, "i'"'] & \mathcal{B}' \arrow[r, "p'"'] & \mathcal{C}' \arrow[r] & 0 \end{tikzcd}$$

(you could be more careful and write $f_j^{\mathcal{A}}, f_j^{\mathcal{B}}, f_j^{\mathcal{C}}$ etc.)

the induced maps on homology form a chain map:

$$\begin{tikzcd} \cdots \arrow[r] & H_n(\mathcal{A}) \arrow[r] \arrow[d, "\alpha_*"'] \arrow[dr, phantom, "\circlearrowright"] & H_n(\mathcal{B}) \arrow[r] \arrow[d, "\beta_*"'] \arrow[dr, phantom, "\circlearrowright"] & H_n(\mathcal{C}) \arrow[r] \arrow[d, "\gamma_*"'] \arrow[dr, phantom, "\circlearrowright"] & H_{n-1}(\mathcal{A}) \arrow[r] \arrow[d, "\alpha_*"'] & \cdots \\ \cdots \arrow[r] & H_n(\mathcal{A}) \arrow[r] & \underbrace{H_n(\mathcal{B})}_{\text{this map is actually } (\beta_*)_n} \arrow[r] & H_n(\mathcal{C}) \arrow[r] & H_{n-1}(\mathcal{A}) \arrow[r] & \cdots \end{tikzcd}$$

This is again proven by diagram chasing (Hatcher pg 127, bottom).

* Rmk on categories + functors.

We'll actually want to show that $\mathcal{F} \circ \mathcal{G}$ is natural:

Pairs of Top spaces, maps of pairs $\xrightarrow{\mathcal{F} \circ \mathcal{G}}$ LESs of $\mathbb{Z}$-mods, chain maps

We already showed maps of spaces ⇝ chain maps (cat of $\mathbb{Z}$-mod)

so squares w/ i_*, j_* commute.

ISTS the square w/ ∂ commutes.

pf

we already have $f_\# \partial = \partial f_\#$ at the chain level

where ∂ is boundary map not connecting map

Let $[\alpha] \in H_n(X, A)$, rep'd by a relative cycle α.

Then

$$f_* \partial [\alpha] = f_* [\partial \alpha] = [f_\# \partial \alpha] = [\partial f_\# \alpha] = \partial f_* [\alpha]$$

connecting map.

by prev. remark ①

//

Equiv. of H^Δ and H

We'll give a sketch. What you need to remember is the method of proof: mainly relies on LES and 5-lemma.

Some setup: Let X be a Δ-complex, let $A \subset X$ be a sub-complex.

① Relative simplicial homology defined analogously; chain groups:

$$\Delta_n(X,A) = \Delta_n(X)/\Delta_n(A)$$

relative simplicial chains

② There is a canonical chain map

$$\varphi^{rel}: \Delta_\bullet(X,A) \longrightarrow C_\bullet(X,A)$$

inducing a canonical homomorphism

$$\varphi^{rel}_*: H^\Delta_\bullet(X,A) \longrightarrow H_\bullet(X,A)$$

Let's write $\varphi^A: \Delta_\bullet(A) \longrightarrow C_\bullet(A)$ &

$$\varphi^X: \Delta_\bullet(X) \longrightarrow C_\bullet(X)$$

thm 2.27 Let (X, A) be a Δ-cpx pair.

The $\varphi_n : H_n^{\Delta}(X,A) \longrightarrow H_n(X,A)$ are isoms for all n

pf.

Case 1 X finite dim'l, and $A = \emptyset$.

Let X^k = the k-skeleton of X, consisting of all simplices of dim k or less.

Commutative diagram: (of parts of LES.)

$$\begin{tikzcd}
H_{n+1}^{\Delta}(X^k, X^{k-1}) \arrow[r] \arrow[d, "\varphi_*^{rel}"'] & H_n^{\Delta}(X^{k-1}) \arrow[r] \arrow[d, "\varphi_*^X"] & H_n^{\Delta}(X^k) \arrow[r] \arrow[d, "\varphi_*^X"] & H_n^{\Delta}(X^k, X^{k-1}) \arrow[r] \arrow[d, "\varphi_*^{rel}"'] & H_{n-1}^{\Delta}(X^{k-1}) \arrow[d, "\varphi_*^X"] \\
H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1})
\end{tikzcd}$$

Step 1 Show φ_k^{rel} in the diag are $\cong$.

Key $\Delta_n(X^k, X^{k-1}) = 0$ when $n \neq k$

$\Delta_k(X^k, X^{k-1}) = \mathbb{Z}\langle k\text{-simplicies}\rangle$

Same for $H_n^\Delta(X^k, X^{k-1})$.

Show same for $H_n(X^k, X^{k-1})$ by thinking about rel cycles.

Step 2 (Base) $H_n^\Delta(X^0) \longrightarrow H_{n-1}(X^0)$

By induction may assume 2nd and 5th maps are isoms

Step 3 Use 5-lemma.

//

Case 2 X is infinite dim

need to use fact that a cpct $C \subset X$ intersects only finitely many simplices

need to use top arguments to show φ_* is surj & inj.

Case 3 Relative

Use LES.

//

More importantly:

From Case 1 we see that $H_n(X)$ is finitely generated when $\Delta_n(X)$ is f.g.

f.g.

F.g. module over $\mathbb{Z}$ (PID) are isom to $\mathbb{Z}^r \oplus \bigoplus_{i=1}^{t} \mathbb{Z}/n_i\mathbb{Z}$

The rank (r) of $H_n(X)$ is called the nth Betti number of X

The n_i are the torsion coefficients.