MAT 150A Fall 2023
Instructor: Melissa Zhang
Exam 1

By providing my signature below I acknowledge that I abide by the University's academic honesty policy. This is my work, and I did not get any help from anyone else:

Name (sign): ____________________________ Name (print): ____________________________

Name of left neighbor: ________________ Name of right neighbor: ________________

If you are next to the wall, then write "Wally" as your left or right neighbor. Write "Nemo" for your left/right neighbor if you don't have a left/right neighbor, respectively.

Question	Points	Score
Q1	20	
Q2	15	
Q3	25	
Total:	60	

- This is a **closed-book** exam. You may not use the textbook, cheat sheets, notes, or any other outside material. No calculators, computers, phones, or any other electronics are allowed.
- The last page of the exam packet is provided for scratchwork. **Do not detach** this sheet from your exam packet.
- You have **45 minutes** to complete this exam. If you are done early, you may leave after handing in your exam packet.
- Everyone must work on their own exam. Any suspicions of collaboration, copying, or otherwise violating the Student Code of Conduct will be forwarded to the Student Judicial Board.
- **This is a proof-based course.** All statements must be justified and argued in the style of a mathematical proof. Failure to do so will result in the loss of correctness and/or style points.

1. Let G be a group. Prove that the map $\varphi : G \to G$, $x \mapsto x^2$, is an endomorphism of G if and only if G is abelian.

SOLUTION.

($\Rightarrow$) Suppose $\varphi(x) = x^2$ is an endomorphism of G. We want to show that any two elements $x, y \in G$ commute, i.e. $xy = yx$.

Since φ is a homomorphism, we have

$$(xy)^2 = \varphi(xy) = \varphi(x)\varphi(y) = x^2y^2,$$

i.e. $xyxy = xxyy$. By cancelling the x on the left and the y on the right, we have $yx = xy$, which is what we wanted to show.

($\Leftarrow$) Now suppose G is abelian. Then

$$\varphi(xy) = (xy)^2 = xyxy = x(yx)y = x(xy)y = x^2y^2 = \varphi(x)\varphi(y),$$

so $\varphi : G \to G$ is a homomorphism, and therefore an endomorphism of G.

2. Let $\varphi : G \to G'$ be a group homomorphism. Suppose that $|G| = 18$ and $|G'| = 15$, and that φ is not the trivial homomorphism. What is the $|\ker\varphi|$?

Solution.

Since $|G| = |\ker\varphi| \cdot |\operatorname{im}\varphi|$, both $|\ker\varphi|$ and $|\operatorname{im}\varphi|$ must divide $|G| = 18$. Since $\operatorname{im}\varphi$ is a subgroup of G', so $|\operatorname{im}\varphi|$ must also divide 15. This means that $|\operatorname{im}\varphi| \in \{1, 2, 3, 6, 18\} \cap \{1, 3, 5, 15\} = \{1, 3\}$.

But φ is not the trivial homomorphism, so $|\operatorname{im}\varphi| > 1$. Therefore $|\operatorname{im}\varphi| = 3$, and so $|\ker\varphi| = 18/3 = 6$.

3. Let H and K be subgroups of G.

   (a) Prove that if $HK = KH$, then HK is a subgroup of G.

   (b) Prove that if H and K are both *normal* subgroups of G, then their intersection $H \cap K$ is also a *normal* subgroup of G.

SOLUTION.

(a) Assume that $HK = KH$. To see that $HK \leq G$, we check *identity*, *inverses*, and *closure*.
(Identity) Since $H, K \leq G$, $1 \in H$ and $1 \in K$; therefore $1 = 1 \cdot 1 \in HK$.
(Inverses) Let hk be an arbitrary element of HK, where $h \in H$ and $k \in K$. Then $(hk)^{-1} = k^{-1}h^{-1} \in KH = HK$, so HK is closed under taking inverses.
(Closure) Now let $h_1, h_2 \in H$, and $k_1, k_2 \in K$, so that h_1k_1, h_2k_2 are arbitrary elements of HK. We want to show that $(h_1k_1)(h_2k_2) \in HK$. Since $k_1h_2 \in KH = HK$, there exist $h' \in H$ and $k' \in K$ such that $k_1h_2 = h'k'$. Then

$$(h_1k_1)(h_2k_2) = h_1(k_1h_2)k_2 = h_1(h'k')k_2 = (h_1h')(k'k_2).$$

Since H and K are both subgroups, $h_1h' \in H$ and $k'k_2 \in K$; therefore $(h_1h')(k'k_2) \in HK$, as we wanted to show.

(b) Let $x \in H \cap K$ and $g \in G$. Since $H \trianglelefteq G$, we have $gxg^{-1} \in H$; similarly, since $K \trianglelefteq G$, we have $gxg^{-1} \in K$. Therefore $gxg^{-1} \in H \cap K$, so $H \cap K \trianglelefteq G$.

Scratchwork

Nothing on this page will be graded.