# MAT 150A HW02

[ADD YOUR NAME HERE]

Due Tuesday, 10/17/23 at 11:59 pm on Gradescope

***Reminder.*** *Your homework submission must be typed up in full sentences, with proper mathematical formatting. The following resources may be useful as you learn to use TeX and Overleaf:*

- *Overleaf's introduction to LaTeX:*
  *https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes*
- *Detexify:*
  *https://detexify.kirelabs.org/classify.html*

**Covered in this HW** Parts of Chp. 2 and 3, esp. §2.2–2.5, 3.2–3.3, 11.1. Groups, abelian groups, order of a group or element of a group, cyclic groups, subgroups; definition of rings, fields, and vector spaces; homomorphisms, normal subgroups, generators and relations.

**Grading** Some of the (parts of) problems will be graded in detail out of several points, and necessary feedback will be given. The rest will be graded out of 2 points. I will reveal which problems are fully graded in the solutions, which will be posted on the Friday following the due date.

## Exercise 1

Let $\operatorname{ord}(g) \in \mathbb{N} \cup \{\infty\}$ denote the order of an element $g$ in a group $G$. Recall that $\operatorname{ord}(g) = |\langle g \rangle|$ or, equivalently,

$$\operatorname{ord}(g) = \min \{n \in \mathbb{N} \ : \ g^n = 1\}.$$

Let $a, b \in G$. Prove that $\operatorname{ord}(ab) = \operatorname{ord}(ba)$.

Solution.

## Exercise 2

Show by example that the product of elements of finite order in a group need not have finite order. What if the group is abelian?

Solution.

## Exercise 3

Prove that every subgroup of a cyclic group is cyclic. *Hint: Work with exponents and use the description of the subgroups of* $\mathbb{Z}^+$.

Solution.

## Exercise 4

Let $U$ denote the group of invertible upper triangular $2 \times 2$ matrices

$$\left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \;:\; a, b, d \in \mathbb{R},\; ad \neq 0 \right\} \subset GL_n(\mathbb{R})$$

and let $\varphi : U \to \mathbb{R}^\times$ be the map that sends $A \mapsto a^2$. Prove that $\varphi$ is a homomorphism, and determine its kernel and image.

Solution.

## Exercise 5

Let $f : \mathbb{R}^+ \to \mathbb{C}^\times$ be the map $f(x) = e^{ix}$. Prove that $f$ is a homomorphism, and determine its kernel and image.

Solution.